A 2D TQFT approach to topological graph polynomials and graphs in thickened surfaces

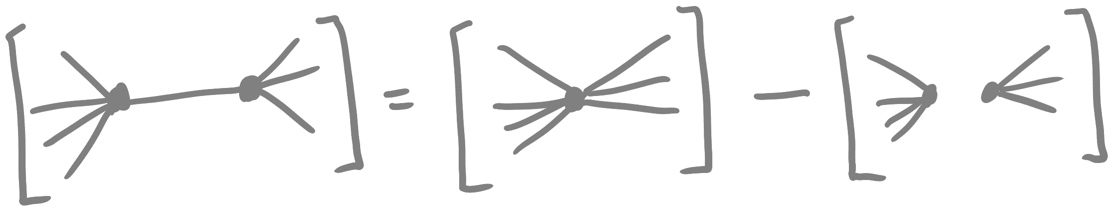
Kyle Miller – University of California, Berkeley
kmill@berkeley.edu

Wednesday, January 6, 2021

Overview

The Yamada polynomial [Yamada '89] is a quantum invariant of spatial graphs.

It satisfies a contraction-deletion relation like classic graph polynomials,



- chromatic polynomial
- flow polynomial
- Tutte polynomial
- Bollobás–Riordan polynomial

For planar graphs, it "is" the flow polynomial.

Extensions of flow polynomial to nonplanar graphs yield invariants of virtual spatial graphs:

[Fleming and Mellor '07], [McPhail-Snyder and M. '18], [Deng, Jin, and Kauffman '18]

Overview

These are relatively easy-to-compute invariants of virtual spatial graphs

- Can we find all extensions of the Yamada polynomial to virtual spatial graph invariants?

The arrow polynomial [Dye and Kauffman '09] is an invariant of virtual knots that incorporates more topological information.

- Can similar information be incorporated into a virtual Yamada polynomial?

Virtual spatial graphs

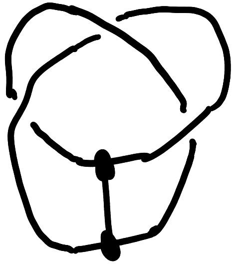
A ribbon graph is a "thickened" graph.

vertices $\rightsquigarrow$ disks

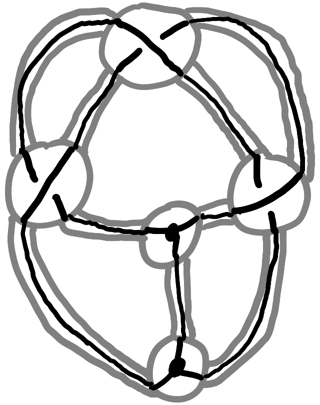
edges $\rightsquigarrow$ thin rectangular strips

A spatial graph is a ribbon graph embedded in S^3 (equivalently, in $S^2 \times [0,1]$)

A spatial graph diagram is a planar graph with special degree-four vertices for crossings:



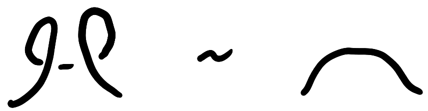
as a planar ribbon graph:



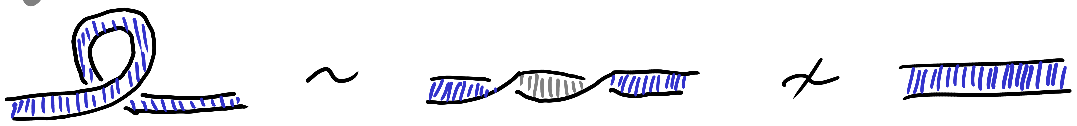
Virtual spatial graphs

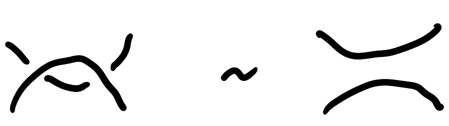
Spatial graphs / isotopy $\approx$ spatial graph diagrams / "Reidemeister moves"

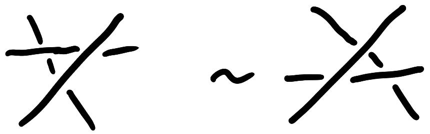
Moves:

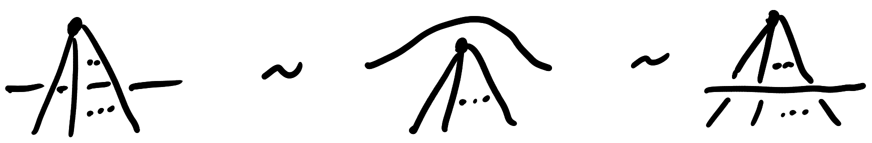
I) 

← Regular isotopy due to strips:



II) 

III) 

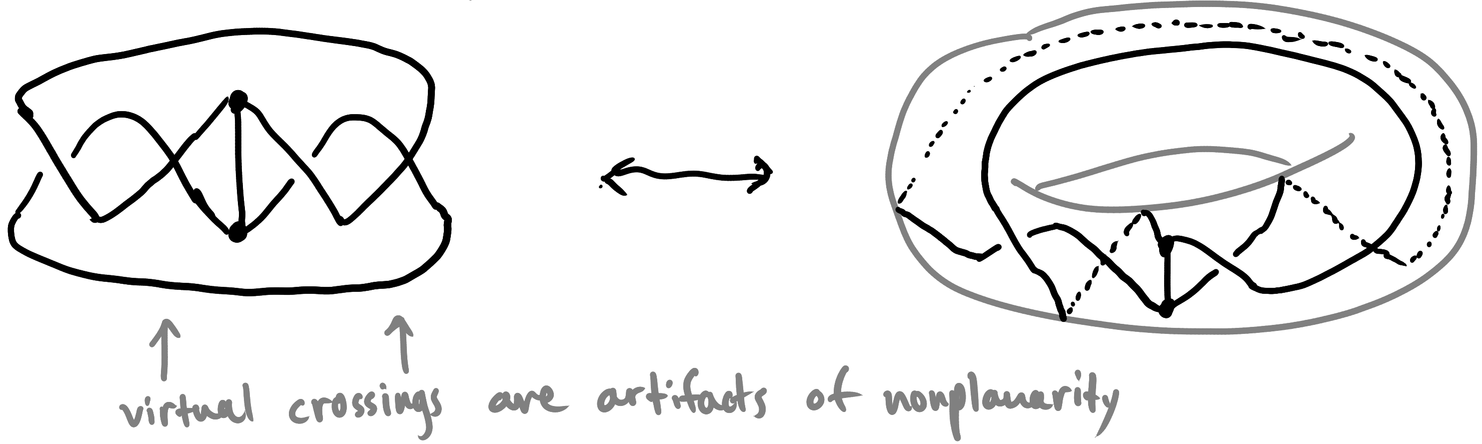
IV) 

Virtual spatial graphs

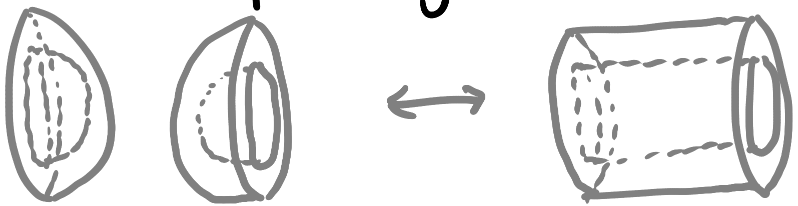
A virtual spatial graph is a spatial graph diagram without regard for planarity!

Equivalence is still via Reidemeister moves.

Ex



[Carter, Kamada, and Saito '02], [Kuperberg '03] Virtual spatial graphs are ribbon graphs in thickened surfaces, modulo stabilization :



(away from graph)

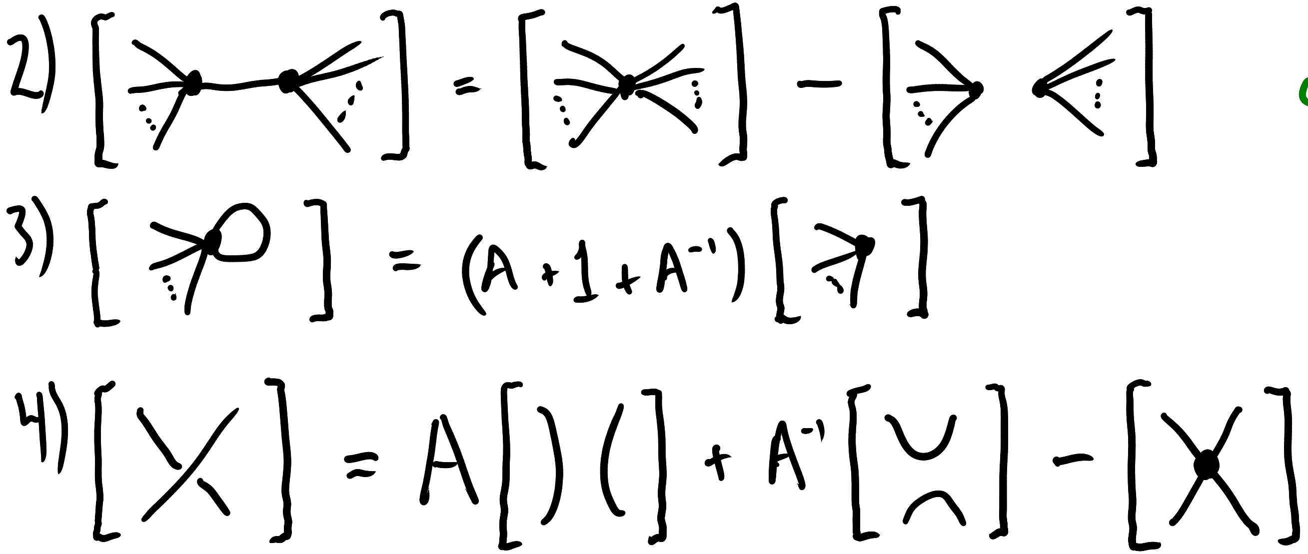
Thm Can always maximally destabilize then isotope.

The Yamada polynomial

G - spatial graph

$R(G;A) \in \mathbb{Z}[A^{\pm 1}]$ is Yamada polynomial, defined by the skein relations

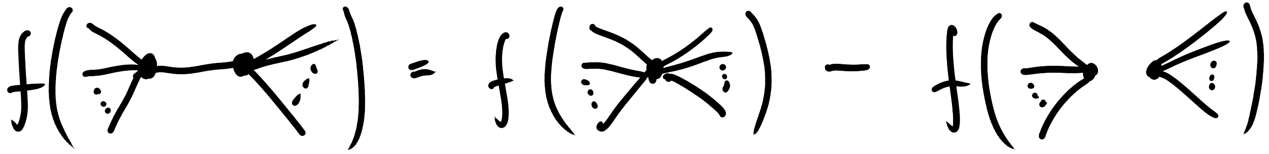
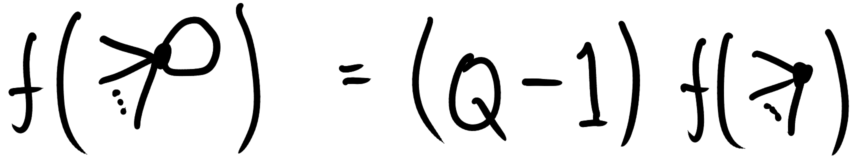
1) $[G \sqcup \bullet] = [G]$



contraction - deletion relation

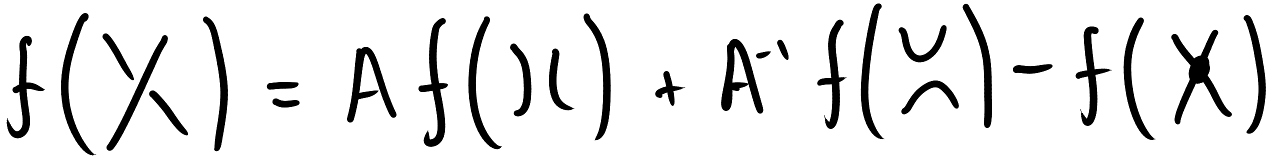
Virtual Yamada polynomials

An extension of R to virtual spatial graphs corresponds to a ribbon graph invariant f satisfying

1) $f(G \sqcup \bullet) = f(G)$
2) 
3) 

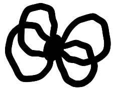
Reduces virtual spatial graphs to interleaved bouquets

ex 

Then: define  with $Q = A + 2 + A^{-1}$

Classification

A virtual Yamada polynomial factors through the ring $\mathcal{R}$ of formal $\mathbb{Z}[A^{\pm 1}]$-linear combinations of virtual spatial graphs modulo the previous relations; disjoint union is multiplication.

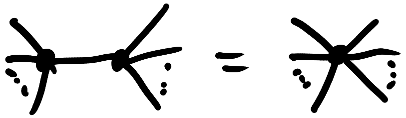
<u>Thm</u> $\mathcal{R} \cong \mathbb{Z}[A^{\pm 1}][$  ,  , $\cdots$, vertex sum of n copies of  , $\ldots]$ (polynomial ring)

<u>Cor</u> A virtual Yamada polynomial is determined by its value on these n-fold vertex sums. (No constraints.)

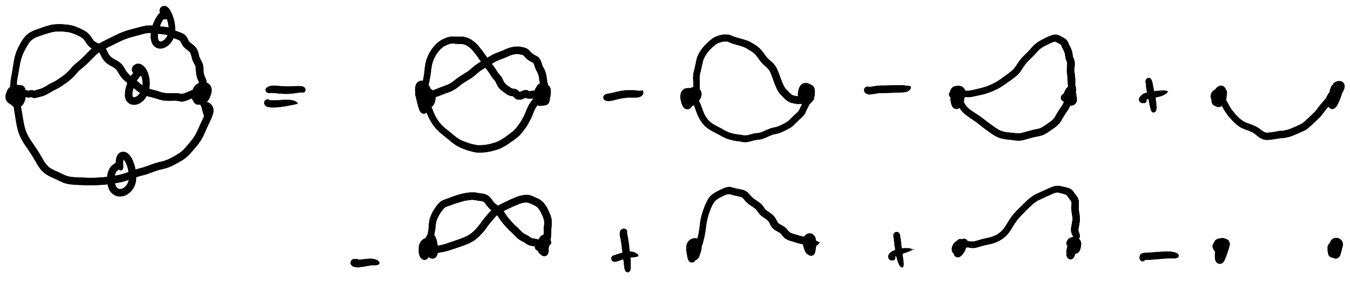
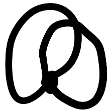
Calculation note

It is easier to consider a variant:

let $\phi := | - \vdots$ be symbol for unexpanded edges, and

i) 

ii) 

ex  $=$  $- Q - Q + 1 - Q + 1 + 1 - 1$

$= 2 - 3Q + X_1 \in \mathbb{Z}[A^{\pm 1}, X_1, X_2, X_3, \dots]$

$(Q = A + 1 + A^{-1})$

(X_g for component with genus g.)

Cor If X_g appears in the polynomial for a virtual spatial graph, then its virtual genus is $\geq g$ (every diagram has genus $\geq g$).

Special cases

[Fleming and Mellor] $X_g = Q^{2g}$

[McPhail-Snyder and M.] $X_g = Q^g$

[Deng, Jin, and Kauffman] essentially $X_g = x^g$, giving an invariant in $\mathbb{Z}[A^{\pm 1}, x]$

Hence it interpolates the above two

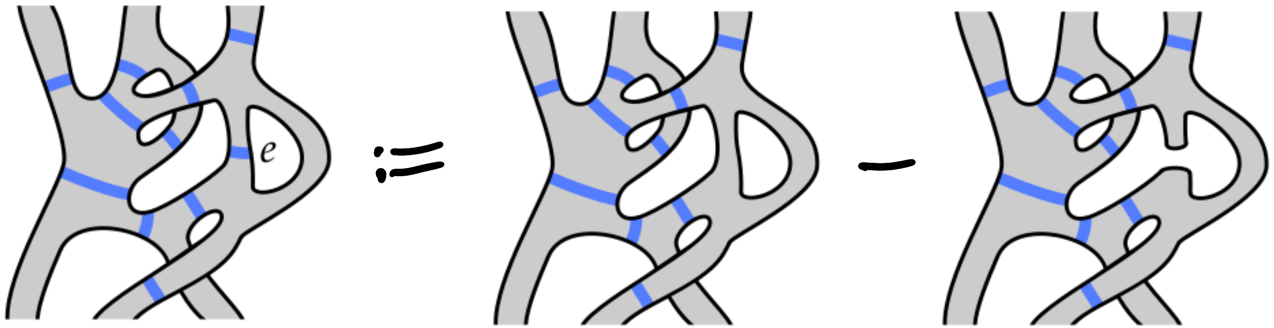
A symmetric monoidal category $2\mathrm{Cob}^{\mathrm{open}}$ [Lauda and Pfeiffer '08]

Objects: disjoint unions of oriented intervals.

Morphisms: "open cobordisms" – surfaces with nonempty boundary

$$\{\text{Symmetric monoidal functors } 2\mathrm{DCob}^{\mathrm{open}} \longrightarrow \mathrm{Vect}\} \cong \{\text{Symmetric Frobenius algebras}\}$$

Extend to have 1D defects defined by



Then get contraction-deletion ribbon graph invariants from functors.

Invariants from symmetric monoidal functors

Thm On ribbon graphs, every such invariant is from a symmetric Frobenius algebra that is a direct sum of some number of the following:

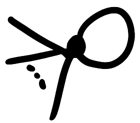
1) $Mat_n(\mathbb{C})$ $\quad \varepsilon : Mat_n \mathbb{C} \to \mathbb{C}$, $\quad A \mapsto c \operatorname{tr} A$ $\quad \Big\}$ $n \geq 0$ in $\mathbb{N}$, $c \neq 0$ in $\mathbb{C}$

2) $\mathbb{C}[x]/(x^n)$ $\quad \varepsilon : \mathbb{C}[x]/(x^n) \to \mathbb{C}$, $\quad x^k \mapsto \begin{cases} 0 & \text{if } k < n-1 \\ c & \text{if } k = n-1 \end{cases}$ $\quad \Big\}$ $n \geq 2$ in $\mathbb{N}$, $c \neq 0$ in $\mathbb{C}$

Thm If the algebra is semisimple (i.e., $A \cong Mat_{n_1}\mathbb{C} \oplus \cdots \oplus Mat_{n_m}\mathbb{C}$) then

$$\underset{\text{Ribbon graph}}{G} \longmapsto \sum_{\substack{H \subseteq G \\ \text{spanning subgraph}}} (-1)^{E(G)-E(H)} \prod_{\substack{F \\ \text{conn. component of } H}} \sum_{i=1}^{m} (c_i^{-1} n_i)^{b_1(F)-1} n_i^{2-2g(F)}$$

Suitability for virtual Yamada polynomials

Need  $= (Q-1)$  . Setting $c_i = Q^{-1} n_i$ works; then

$$G \longmapsto \sum_{H \subseteq G} (-1)^{E(G)-E(G)} Q^{b_1(H)-b_0(H)} \prod_F Y_g \qquad \text{with} \quad Y_g = \sum_{i=1}^{m} n_i^{2-2g}$$

For m and $n_1, \dots, n_m$ generic, $Y_0, Y_1, Y_2, \dots$ are algebraically independent.

<u>Thm</u> The collection of all symmetric monoidal functors $2\mathrm{DCob}^{\mathrm{open}} \to \mathrm{Vect}$ that extend to virtual Yamada polynomials embed $\mathcal{R}$ in $\mathbb{C}[A^{\pm 1}]$-Mod, i.e., if G, G' are virtual spatial graphs with different images in $\mathcal{R}$, then there is such a functor that can distinguish them.

The Bollobás-Riordan polynomial

Using $A = (\mathrm{Mat}_n \mathbb{C})^m$ with $c_1 = \cdots = c_n = c$, then the functor computes (a version of) the Bollobás-Riordan polynomial.

$$A \cong \mathbb{C}^m \otimes \mathrm{Mat}_n \mathbb{C} \cong \mathbb{C}^m \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^n$$

Graphical calculus for functor :

with

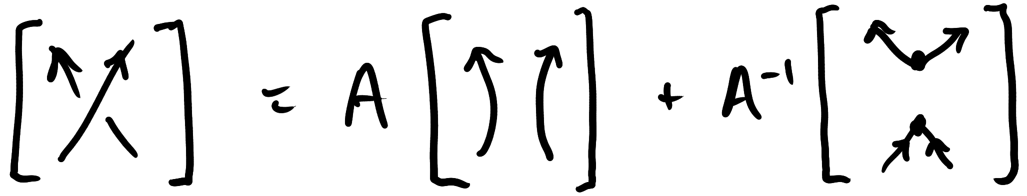
If $cmn = 1$ and $mn^2 = A + 2 + A^{-1}$, get a 2D locus of virtual Yamada polynomials.

Encompasses [Fleming and Mellor '07], [McPhail-Snyder and M. '18], [Deng, Jin, and Kauffman '18]

Note: with $m=1$, $c = -A^2$, $n = -A^2 - A^{-2}$, reduces to rescaled Kauffman bracket [Dasbach, Futer, Kalfagianni '08]

Arrow polynomial [Dye and Kauffman '09]

Kauffman bracket for oriented virtual links.



$\rightsquigarrow$ polynomial in $\mathbb{Z}[A^{\pm 1}, X_1, X_2, \cdots]$

using a rule for net # of cusps.

[M. '20 unpublished] Can compute generalization using a functional on skein module.

$L \subset \Sigma \times [0,1]$ oriented link, there is map $H_1(L) \to H_1(\Sigma)$

given simple closed curve $C \subset \Sigma$, there is bilinear map

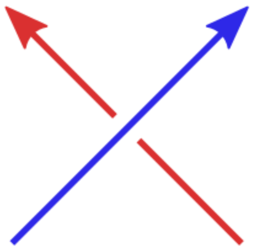
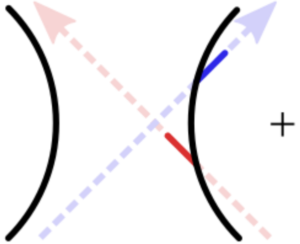
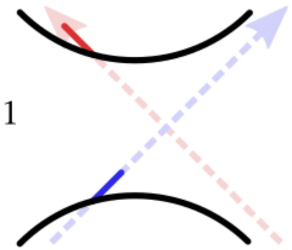
$$H_1(L) \otimes H_1(C) \xrightarrow{\cap} H_0(\Sigma) \xrightarrow{\varepsilon} \mathbb{Z}$$

Fixing an $\alpha \in H_1(C)$, by U.C.T. get element of $H^1(L) \cong H_0(L)$.

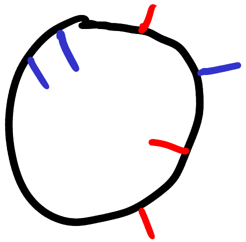
Weight C in Kauffman bracket expansion by $X_{\pm[C]}$ with $\pm[C] \in \pm H_0(L)$. $X_0 := 1$

Arrow polynomial

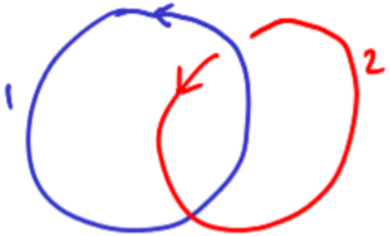
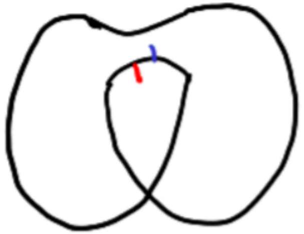
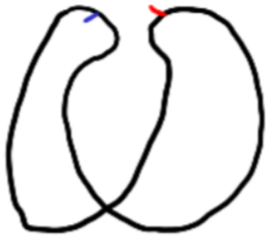
Practically,

 $= A$  $+ A^{-1}$ 

whiskers

ex  $= (-A^2 - A^{-2})\, X_{\pm(1,-1)}$

ex

 $= A$  $+ A^{-1}$ 

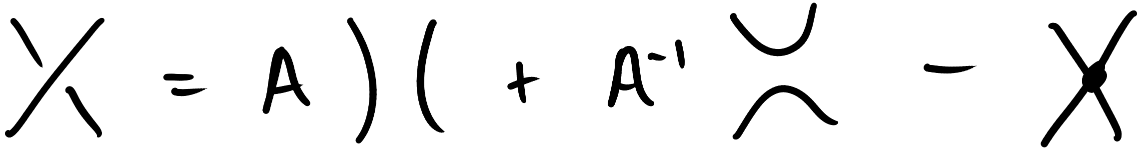
$$= A(-A^2 - A^{-2}) X_{1,-1} + A^{-1}(-A^2 - A^{-2}) X_{1,1} \xrightarrow[\text{usual}]{} A(-A^2 - A^{-2}) + A^{-1}(-A^2 - A^{-2}) X_1$$

Arrow polynomial $X_{(a,b)} \longmapsto X_{|a+b|/2}$

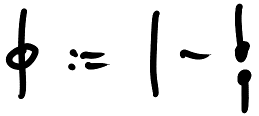
Arrow Yamada polynomial

Consider G a virtual spatial graph, represented as a ribbon graph in $\Sigma \times [0,1]$.

Can expand G in Yamada Skein Module:



(along with contraction-deletion, etc)

Also assume we used  convention.

Result: linear combination of 1-complexes in Σ.

How to put a functional on Yamada Skein Module so that one obtains a virtual spatial graph invariant?

Arrow Yamada polynomial

Rather than just 1-complex $\Gamma \rightsquigarrow X_g$ with $g = \text{genus } \Gamma$ like before, use bilinear form

$$H_1(G) \otimes H_1(\Gamma) \xrightarrow{\cap} H_0(\Sigma) \xrightarrow{\varepsilon} \mathbb{Z}$$

For each connected Γ, take matrix M_Γ of this.

Define $[\Gamma] = M_\Gamma \cdot GL(H_1(\Gamma))$, the coset of action on matrices.
(so it doesn't depend on basis of $H_1(\Gamma)$)

Then can weight component Γ of expansion by $X_g Y_{M_\Gamma \cdot GL(H_1(\Gamma))}$

If we do not have a preferred basis for $H_1(G)$, can do double cosets

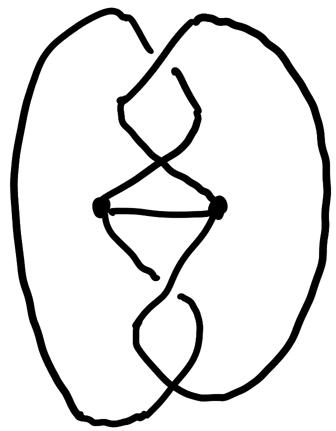
$$GL(H_1(G)) \cdot M_\Gamma \cdot GL(H_1(\Gamma)) \rightsquigarrow \text{Smith normal form}$$

get sequence $d_1, d_2, d_3, \ldots$ with $d_i \mid d_{i+1}$

hence get variables $Y_{d_1, d_2, \ldots}$

Arrow Yamada polynomial

ex



$$-\frac{(-1+A)^3\,(1+A)^2}{A^3}+\frac{(-1+A)\,(1+A)^2\,\left(4+3A+2A^2+3A^3+4A^4\right)\,Y[1]}{A^4}-\frac{(-1+A)\,(1+A)^4\,\left(1+A+A^2+A^3+A^4\right)\,Y[1]^2}{A^5}-\frac{2\,(-1+A)\,(1+A)^2\,Y[2]}{A^2}-\frac{(-1+A)^3\,(1+A)^4\,X[1]\times Y[1,2]}{A^4}$$

Both Y and X terms indicate positive virtual genus.

Thanks!