Invariants of graphs in thickened surfaces from topological graph polynomials

Rice topology seminar 11/18/2019

Kyle Miller, UC Berkeley

* Overview

- Ribbon graph invariants
  - ~~Algebraic~~ Graphical/TQFT ways to calculate them
  - The Yamada polynomial for spatial graphs
  - A systematic extension to ~~[illegible]~~ ribbon graphs in thickened surfaces (modulo stabilization)
  - Towards invariants of surface graphs

* Ribbon graphs

<u>Def</u> A <u>ribbon graph</u> is a graph Γ embedded in the interior of a compact oriented surface Σ s.t. $\Gamma \hookrightarrow \Sigma$ is a homotopy equivalence.

<u>ex</u>  vs 

The data of a ribbon graph is the "rotation system" of half edges around vertices

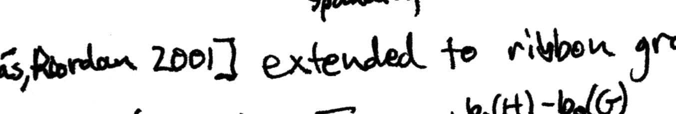
[Tutte 1954] defined graph invariant

$$T_G(x,y) = \sum_{\substack{H \subseteq G \\ \text{spanning}}} (x-1)^{b_0(H)-b_0(G)} (y-1)^{b_1(H)}$$

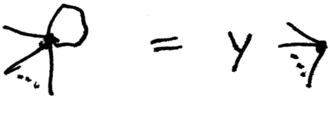
[Bollobás, Riordan 2001] extended to ribbon graphs

$$BR_G(x,y,z) = \sum_{H \subseteq G} (x-1)^{b_0(H)-b_0(G)} (y-1)^{b_1(H)} z^{2g(H)}$$

These satisfy "skein relations" like

(1) 

non-bridge contraction deletion

(2) 

(3) $[G_1 \sqcup G_2] = [G_1] \cdot [G_2]$

Many functions on graphs (like chromatic poly) are specializations

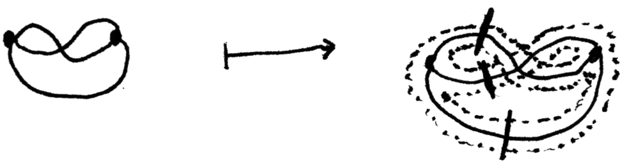
[Thistlethwaite 1987] The Jones poly of an alternating link is a specialization of T of the Tait/checkerboard graph

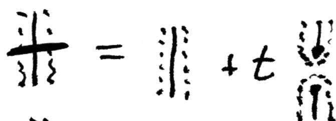
[Dasbach, et al. 2008] The Jones poly of a link is a specialization of BR of the Turaev ribbon graph.

Is there a "visually obvious" proof of these?

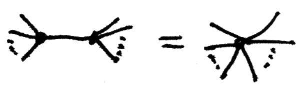
* BR graphically

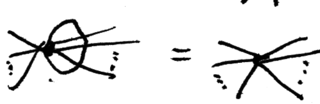
Ribbon graph $\rightsquigarrow$ trace diagram



where • 

•  $= n$ (like TL)

• 

• 

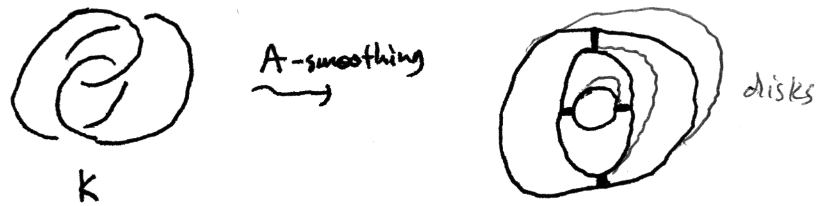
calculates m^{b_0}

• $\bullet = m$

Get poly in $\mathbb{Z}[t, m, n]$, equiv. to BR:

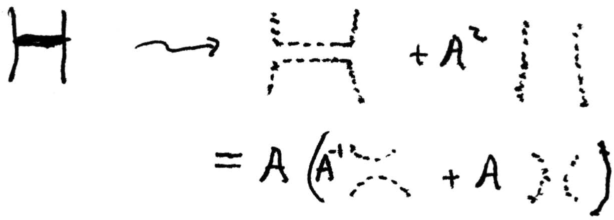
is $t^{b_1(G)} (mn)^{b_0(G)} \, BR_G(tmn+1, t^{-1}n+1, n^{-2})$

Turaev Ribbon graph:



$\mathcal{J}_K \subseteq B^3$

If $m=1$, $t=A^2$, $n=-A^2-A^{-2}$,



hence $A^{-\text{crossings}} BR_{\mathcal{J}_K}(m=1, t=A^2, n=-A^2-A^{-2})$

$= (-A^2-A^{-2}) \langle K \rangle_A$

Kauffman bracket.

* Yamada polynomial (1989)

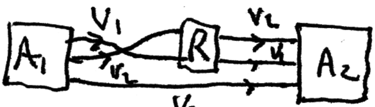
Given a spatial graph, a ribbon graph embedded in S^3, 

H a quasitriangular Hopf algebra, (like quantum grp)

for each edge e a repr. V_e, (color)

for each vertex v an element $A_v \in \mathrm{Hom}_H(V_{e_1} \otimes \cdots \otimes V_{e_k}, V_{e_1'} \otimes \cdots \otimes V_{e_l'})$

the Reshetikhin-Turaev invariant is



Many choices!

~~Sometimes few choices, and somehow~~

loops $= -q - q^{-1}$ $\quad \times = q^{1/2})(+ q^{-1/2} \genfrac{}{}{0pt}{}{\cup}{\cap}$

$H = U_q(sl(2))$ (think Temperley-Lieb & Kauffman bracket)

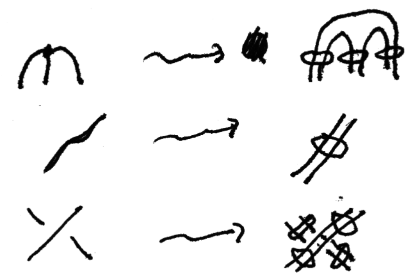
all edges colored by V_2, a 3-dim irred. repr.

Turns out $\dim(\mathrm{Hom}_H(V_2^{\otimes 3}, V_0)) = 1$, so trivalent vertices have single choice.

$\oplus := || + (q+q^{-1})^{-1} \genfrac{}{}{0pt}{}{\cup}{\cap}$ is 2nd Jones-Wenzl projector, $V_1 \otimes V_1 \to V_1 \otimes V_1$

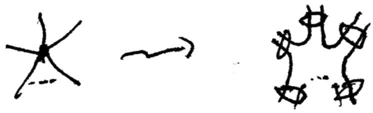
$\mathrm{im}\, \oplus \cong V_2$

Let

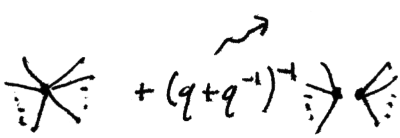


Gives a $\mathbb{Z}[q^{\pm 1}]$-valued invariant of trivalent spatial graphs.

But, say

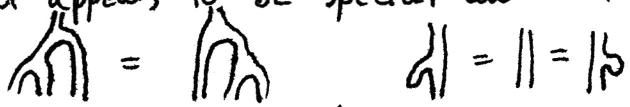


Then



So has well-defined contraction-deletion.

What appears to be special about $V_1^{\otimes 2}$:



associativity of (multiplication) ; U is a unit ; and (comultiplication), ∩

compatibility of (multiplication) & (comultiplication):

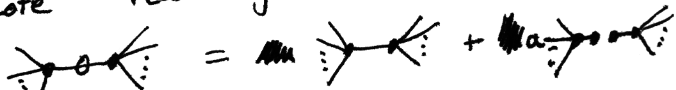
This makes $V_1^{\otimes 2}$ a <u>Frobenius algebra</u>

Idea: Frobenius algebras underlie contraction-deletion invariants.

Let's analyze all such ribbon graph invariants then apply it to extend Yamada poly to virtual spatial graphs.

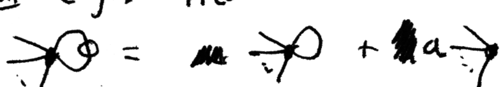
* Ribbon graph invt analysis

Let A be Frob. alg. over $\mathbb{C}$.
Let ϕ denote "real edge" and | a contractible edge.

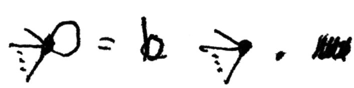


so ϕ = | + a

Suppose <u>all</u> edges treated uniformly, with

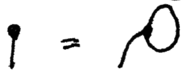


And suppose



Can renormalize so $b = 1$.

This is a strong assumption: (A is <u>special</u>)



$= x \mapsto tr(y \mapsto xy)$ as a trace diagram

and

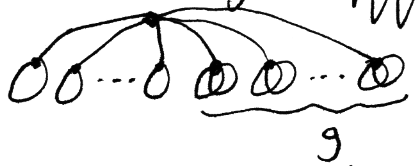
implies is nondegenerate

Thus A is semisimple

Artin-Wedderburn $\Rightarrow$ $A \cong \bigoplus_{i=1}^{N} \underset{\substack{\cup \\ e_i}}{\mathrm{Mat}_{n_i}(\mathbb{C})}$ identity

Can calculate  $= \sum_{i=1}^{N} n_i^{-2} e_i$  $= n_i^2$

Hence conn. ribbon graph

$\rightsquigarrow$ 

$=$  $= \sum_{i=1}^{N} n_i^{2-2g} =: X_g$

$(X_0 = \dim A)$

So, with Γ a ribbon graph, get invt.

$$F_\Gamma^A(a) = \sum_{H \subseteq G} a^{|E(G)| - |E(H)|} \prod_{C \text{ component of } H} X_{g(C)}$$

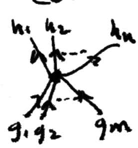
Can just define $F_\Gamma(a, X) = \sum_{H \subseteq G} a^{|E| - |E(H)|} \prod_{C \text{ comp of } H} X_{g(C)}$

in $\mathbb{Z}[a, X_0, X_1, \cdots]$.

<u>Special cases</u>

- $\mathbb{C}[G]$, G finite group.

$|G|^{|E|-|V|} F_\Gamma^{\mathbb{C}[G]}(-|G|^{-1})$ counts "nowhere-1 G flows"



$g_1 \cdots g_m = h_1 \cdots h_n$

$\forall i,j,\ g_i \neq 1,\ h_j \neq 1$

G abelian: flow polynomial ($X_g = |G|$)

(For planar graphs, $F_\Gamma^{\mathbb{C}[\mathbb{Z}/4\mathbb{Z}]}(-1/4) \neq 0$ $\Leftrightarrow$ 4-color theorem!)

• $A = \mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \otimes \mathrm{Mat}_n(\mathbb{C}) \cong \bigoplus_{i=1}^{m} \mathrm{Mat}_n(\mathbb{C})$ $(\chi_g = mn^{2-2g})$

$F^A_\Gamma(a)$ gives BR poly from earlier.

* Virtual spatial graphs

<u>def</u> [Carter, Kamada, Saito 2002; Kuperberg 2003]
A <u>virtual spatial graph</u> is a ribbon graph in a thickened closed ori. sfc, modulo surgery on vertical annuli in the complement:

<u>Thm</u> [Kup.] VSGs have unique representatives ~~without essential vertical~~ in minimal genus thickened sfc.

<u>Cor</u> Virtually equiv spatial graphs (in $S^2 \times I$) are isotopic.

to next *
Yamada poly has been extended to VSGs:
[Fleming, Mellor, 2007] with flow poly
[McPhail-Snyder, M., 2018] with "S-poly"
[Deng, Jin, Kauffman, 2018] by skein rel. solving
[M.] with B-R poly

Convenient formulation of Yamada poly

$R_\Gamma(A) \in \mathbb{Z}[A^{\pm 1}]$

with 1) $R(\;) = R(\;) - R(\;)$

2) $R(\;) = (A+1+A^{-1})\, R(\;)$

3) $R(\Gamma \amalg \bullet) = R(\Gamma)$

4) $R(\;) = A\,R(\;) + A^{-1} R(\;) - R(\;)$

Any ribbon graph invariant satisfying (1-3) gives extension to VSGs

[go back to cites]

If implemented with Fr. alg., (2) $\Rightarrow$ is special, so is explained by $F_\Gamma(a, \underline{X})$ ~~[illegible]~~

(1) and (3) $\Rightarrow$ invt is $R^0_\Gamma(A,\underline{X}) = \sum_{S \subseteq E(G)} (-1)^{|S|} X_0^{b_1(G-S)} \prod_{c \in \pi_0(G-S)} (X_{g(c)} X_0^{-1})$

(2) $\Rightarrow$ $X_0 = (-A^{1/2} - A^{-1/2})^2 = A + 2 + A^{-1}$

def $\overline{R}_\Gamma(A, \underline{X}) \in \mathbb{C}(A)[X_1, X_2, \cdots]$ for Γ a VSG

is • if no crossings,

$\overline{R}_\Gamma(A, \underline{X}) =$ ~~[illegible]~~ $R^0_\Gamma(A, \underline{X})$ with $X_0 = A + 2 + A^{-1}$

• $\overline{R}(\;) = A\,\overline{R}(\;) + A^{-1}\overline{R}(\;) - R(\;)$

Prop If Γ a VSG and $\overline{R}_\Gamma(A, \underline{X}) \notin \mathbb{C}(A)$, then Γ is not equivalent to an $S^2 \times I$ spatial graph.

Generalizes [Miyazawa 2006], essentially is $R^{\mathbb{C}[\mathbb{Z}/n^2\mathbb{Z}]} \overset{?}{=} R^{\mathrm{Mat}_n(\mathbb{C})}$ for virtual links.

counterex $\overline{R}\Big($  $\Big) \in \mathbb{C}(A)$ but not classical!

* Some TQFT with $F^A(a, \underline{X})$

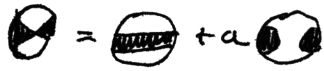
$\boxed{\Gamma_1}\!-\; = p_1 \bullet\!- \quad$ so $\quad \boxed{\Gamma_1}\!-\!\bullet = p_1 X_0$.

$\boxed{\Gamma_1}\!-\!\boxed{\Gamma_2} = p_1 p_2 \bullet\!-\!\bullet = p_1 p_2 X_0 = X_0^{-1} \boxed{\Gamma_1}\!-\!\bullet \;\; \bullet\!-\!\boxed{\Gamma_2}$

2.10

* Beyond ribbon graphs

Medial construction

Contraction-deletion analogue: 

Reduces to ~~graphs~~ surfaces w/ B&W partition

There is 2-cat. 2DCob of such sfcs
("nonplanar planar algebras")

Want to characterize functors
2DCob $\longrightarrow$ $\mathrm{Bim}_{\mathbb{C}}$

ex black regions $\rightsquigarrow$ $\mathbb{C}[\mathbb{Z}/x\mathbb{Z}]$
white regions $\rightsquigarrow$ $\mathbb{C}[\mathbb{Z}/y\mathbb{Z}]$
boundaries $\longrightarrow$ $\mathbb{C}^{z}$ or $(\mathbb{C}^{z})^*$

gives $\displaystyle\sum_{H\subseteq G} a^{|E|-|E(H)|}\, x^{b_0(H)}\, y^{b_0^{\perp}(H)}\, z^{b_0(\partial H)}$

which is equiv. to Krushkal poly (2011)
($y=1 \rightsquigarrow$ BR)