3-manifold seminar

2/12/18
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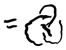
The arithmeticity of figure eight knot orbifolds

Hilden – Lozano – Montesinos (HLM) 1992

<u>Thm 3</u> (HLM) Let $K = 4_1 =$  the figure-eight knot, K_n its n-fold cyclic branched cover, and $(K,n) = K_n/(\mathbb{Z}/n\mathbb{Z})$ be its quotient orbifold (S^3 with $\mathbb{Z}/n\mathbb{Z}$ isotropy along K). Then (K,n) has an arithmetic orbifold fundamental group iff $n = 4,5,6,8,12,\infty$, as a subgroup of $PSL(2,\mathbb{C})$.

<u>Rmk</u> (K,∞) denotes K as cusp in S^3, which is $S^3 - K$ with complete hyp. metric.
Reid 1991: For K a hyperbolic knot, $\pi_1(S^3-K)$ is arithmetic iff $K = 4_1$.

* Motivation

HLM in the 80's studied Thurston's question of links $L \subset S^3$ that are <u>universal</u> — those for which every closed ori 3-mfld is a branched cover over S^3 with L as the branch locus.

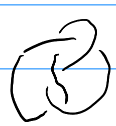
Thurston:  is hyp. orbifold (with group called $B(4,4,4)$)

HLM: this is universal, → and actually factors through this orbifold and so are all 2-bridge non-torus links (includes 4_1)

W. Neumann & Reid: $B(4,4,4)$ is arithmetic, so branched covers of this have arithmetic groups
$\Rightarrow$ every cpct ori 3-mfld is mfld cover of S^3 branched over Borromean rings

<u>Question</u> Which hyperbolic orbifolds are arithmetic?
HLM studied cyclic branched covers of Borr. rings & 4_1.

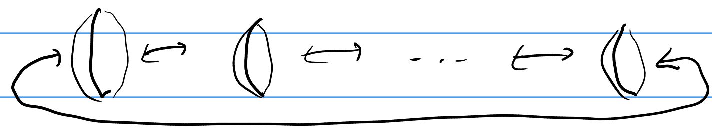
* Branched covers of the figure-8 knot

$K =$  $=$ 

Murasugi sum of two Hopf links
$\Rightarrow$ K is <u>fibered knot</u>
i.e., S^3 has open book decomp. with binding K

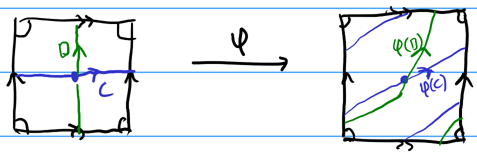
↑ can see oriented genus-1 sfc bounding K. This Seifert sfc is a page.

Let $\Sigma \subset S^3$ be this genus -1 sfc w/ ∂, $\varphi : \Sigma \to \Sigma$ the monodromy from front to back. The n-fold cyclic branched cover K_n of K is from taking n copies of $S^3 - \Sigma \cong \Sigma \times I / (\partial\Sigma, s) \sim (\partial\Sigma, t)$ and gluing together using φ.



This has natural $\mathbb{Z}/n\mathbb{Z}$ action. Quotient is S^3 orbifold with K as branch locus w/ $\mathbb{Z}/n\mathbb{Z}$ isotropy gp., call it (K, n).

Σ:



$\varphi(C) = CDC$

$\varphi(D) = DC$

Can use monodromy to compute $\pi_1(S^3 - K)$ as HNN extension ($S^3 - K$ as mapping torus)

$$\pi_1(S^3 - K) = \langle \mu, C, D \mid \mu CDC \mu^{-1} = C, \ \mu DC \mu^{-1} = D \rangle$$

Also to compute $\pi_1(K_n)$. Let C_i, D_i be loops in i^{th} copy.

Then
$$\pi_1(K_n) = \langle \{C_i, D_i\} \mid C_i D_i C_i = C_{i-1}, \ D_i C_i = D_{i-1} \rangle$$
indices mod n

$$= \langle \{C_i, D_i\} \mid C_i D_{i-1} = C_{i-1}, \ D_i C_i = D_{i-1} \rangle$$

note: $D_n \to X_1$, $C_n \to X_2$, $D_{n-1} \to X_3$, $C_{n-1} \to X_4$, $\ldots$, $D_1 \to X_{2n-1}$, $C_1 \to X_{2n}$

$$= \langle X_1, \ldots, X_{2n} \mid X_i X_{i+1} = X_{i+2} \rangle$$
indices mod $2n$

$$= F(2, 2n), \text{ a } \underline{\text{Fibonacci group}}$$

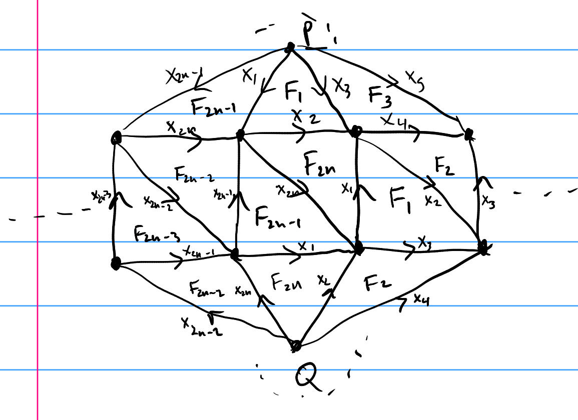
only finite examples

m	$F(2,m)$
1	1
2	1
3	Q_8
4	$\mathbb{Z}/5$
5	$\mathbb{Z}/11$
7	$\mathbb{Z}/29$

* $F(2,2n)$

1998

Helling-Kim-Mennicke (HKM) present $F(2,2n)$ as π_1(closed ori 3-mfld) by face identification of a B^3 polyhedron. Let M_n be this polyhedron quotient: $(n \geq 1)$

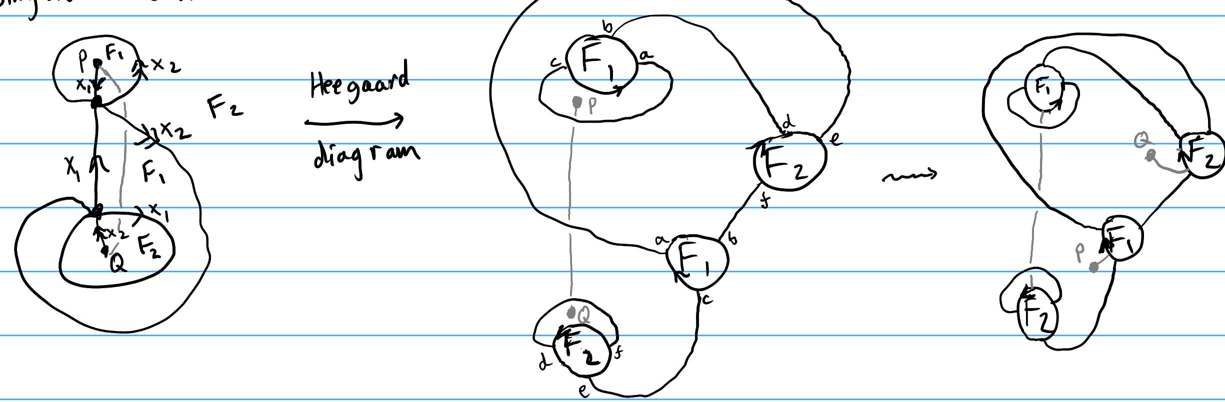


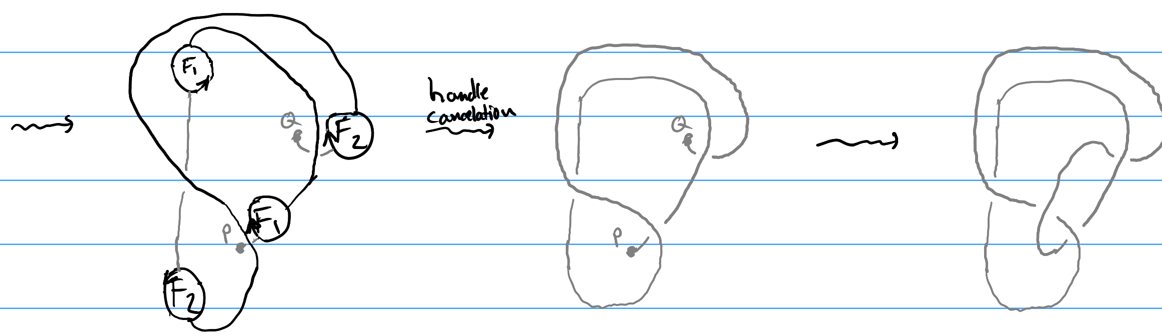
ex M_5 is from icosahedron

$\left.\begin{array}{l} 1 \text{ vtx} \\ 2n \text{ edges} \\ 2n \text{ faces} \\ 1 \text{ solid} \end{array}\right\}$ Euler char $= 0$

Prop (Seifert-Threlfall (1934)) Let K be a 3-D clsd ori pseudomfld from identifying ∂ faces of a simply conn. polyhedron. K is a mfld if $\chi(K) = 0$.

$\mathbb{Z}/n\mathbb{Z}$ acts on M_n by $F_i \rightsquigarrow F_{i+2}$. Quotient is M_1 with interior path PQ the singular locus.





$= 4_1$

So (K, n) is $M_n/(\mathbb{Z}/n\mathbb{Z})$ (HLM & J. Howie)

__Thm__ (HKM) $M_1 \cong S^3$, $M_2 \cong L(5,2)$, M_3 is Euclidean, $M_{n\geq 4}$ is hyperbolic.
They show for $n\geq 4$ by tessellating $\mathbb{H}^3$ with polyhedron.

Hence there are discrete representations $\omega_n : \pi_1(S^3-K) \longrightarrow PSL(2,\mathbb{C}) \subset \mathrm{Isom}^+ \mathbb{H}^3$ for $n\geq 4$ such that (K,n) is $\mathbb{H}^3 / \omega_n(\pi_1(S^3-K))$

* $PSL(2,\mathbb{C})$ reprs of knot groups

Let $K\subset S^3$ be knot and $\omega : \pi_1(S^3-K) \longrightarrow PSL(2,\mathbb{C})$. Pullback:

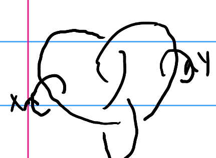
$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & G \arrow[r, "q'"] \arrow[d, "\omega'"] & \pi_1(S^3-K) \arrow[r] \arrow[d, "\omega"] & 1 \\
1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & SL(2,\mathbb{C}) \arrow[r, "q"] & PSL(2,\mathbb{C}) \arrow[r] & 1
\end{tikzcd}$$

G is $\mathbb{Z}/2\mathbb{Z}$ central ext'n of $\pi_1(S^3-K)$.

Classified by $H^2(\pi_1(S^3-K);\mathbb{Z}/2\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Sphere thm} \\ \Rightarrow K(\pi,1)}}{\cong} H^2(S^3-K;\mathbb{Z}/2\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Alexander} \\ \text{duality}}}{\cong} \tilde{H}_0(K;\mathbb{Z}/2\mathbb{Z}) = 0$

Hence q' has a section s, and $\omega'\circ s : \pi_1(S^3-K) \to SL(2,\mathbb{C})$ is a lift.
$(G \cong \mathbb{Z}/2\mathbb{Z} \times \pi_1(S^3-K))$

* $SL(2,\mathbb{C})$ reps of "Listing's knot" (4_1) (Whittemore 1973)



Wirtinger: $\pi_1(S^3-K) = \langle x,y \mid x^{-1}yxy^{-1}xyx^{-1}y^{-1}xy^{-1}\rangle =: G$

$\omega : G \longrightarrow SL(2,\mathbb{C})$ nonabelian iff

for $A=\omega(x)$, $B=\omega(y)$,

$$\alpha = \mathrm{tr}\, A = \mathrm{tr}\, B$$

$$\beta = \mathrm{tr}\, AB = \tfrac{1}{2}\left(1+\alpha^2 \pm \sqrt{(\alpha^2-1)(\alpha^2-5)}\right)$$

And up to conj: $(\alpha\neq\pm 2)$

$$A = \begin{bmatrix} \lambda & \\ & \lambda^{-1}\end{bmatrix} \qquad B = \begin{bmatrix} \mu & 1 \\ \mu(\alpha-\mu) & \alpha-\mu \end{bmatrix}$$

$$\lambda = \tfrac{1}{2}\left(\alpha \pm\sqrt{\alpha^2-4}\right) \qquad \mu = \frac{\lambda\beta-\alpha}{\lambda^2-1}$$

$\alpha = \pm 2$:

$$A = \pm\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ e^{i\pi/3} & 1\end{bmatrix}$$

* Hyp. str of (K, n)

Consider $\omega_n : G \longrightarrow PSL(2,\mathbb{C})$ from before, lifted to $SL(2,\mathbb{C})$

A, B are elliptics of order n: $A^n = B^n = -I$ in some lift

Conjugating: $A = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$ with $\lambda^n = -1$.

Can assume $\lambda = e^{\pi i/n}$ after some field automorphism.

$\alpha = \operatorname{tr} A = 2\cos(\pi/n) \in (\sqrt{2}, 2)$ for all $n \geq 4$

Thus $\beta \notin \mathbb{R}$.

<u>Thm</u> (Reid's criterion, 1987)

For $\Gamma \leq SL(2,\mathbb{C})$ of finite covol, let $\Gamma^{(2)} = \langle \{ g^2 : g \in \Gamma \} \rangle$.

Γ is arithmetic if

(i) Trace field $k^{(2)} = \mathbb{Q}(\operatorname{tr}(g) : g \in \Gamma^{(2)})$ is a finite ext'n of $\mathbb{Q}$ with exactly one $\mathbb{C}$-place.

(ii) $\forall g \in \Gamma^{(2)}$, $\operatorname{tr}(g)$ is alg. integer

(iii) $\forall$ emb $\sigma : k^{(2)} \hookrightarrow \mathbb{R}$, $\sigma(\{\operatorname{tr}(g) \mid g \in \Gamma^{(2)}\})$ is bounded.

$\Gamma^{(2)}$ is derived from a quaternion algebra

Let $\Gamma_n \leq SL(2,\mathbb{C})$ be $\omega_n(G)$

$k_n = \operatorname{tr}$ field of Γ_n, $k_n^{(2)} = \operatorname{tr}$ field of $\Gamma_n^{(2)}$

Lemma (HLM2 5.1 & 5.3) gives
"On Borromean orbifolds"

$$k_n = \mathbb{Q}(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \mathbb{Q}(\alpha, \beta)$$

$$k_n^{(2)} = \mathbb{Q}(\operatorname{tr}(A^2), \operatorname{tr}(B^2), \operatorname{tr}(A^2B^2)) = \mathbb{Q}(\alpha^2, \beta)$$

$$= \mathbb{Q}(\cos^2(\pi/n), \Theta_n)$$

where $\Theta_n = (4\cos^2(\pi/n) - 1)(4\cos^2(\pi/n) - 5)$

Prop 1 $k_n^{(2)}$ has exactly one $\mathbb{C}$-place iff $n = 4, 5, 6, 8, 12, \infty$

Pf sketch $\underset{\substack{\| \\ k'}}{\mathbb{Q}(\cos\frac{2\pi}{n})} \hookrightarrow k_n^{(2)}$. Auts of k' extend to auts of $k_n^{(2)}$

$$\cos\frac{2\pi}{n} \mapsto \cos\frac{2\pi j}{n} \qquad (n, j) = 1$$

$$\text{then} \quad \Theta_n \longmapsto \sqrt{(4\cos^2\tfrac{\pi j}{n} - 1)(4\cos^2\tfrac{\pi j}{n} - 5)} = \Theta_n'$$

If $j \in (1, n/3)$ then $\Theta_n' \notin \mathbb{R}$

If $n \geq 4$ and $n \neq 4, 5, 6, 8, 12$, $\exists$ such $j \Rightarrow > 1$ place

Now cases, ex $n = 5$ $k_5^{(2)} = \mathbb{Q}(\Theta_5)$ $\Theta_5 = \sqrt{\frac{-1-3\sqrt{5}}{2}}$ $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$

two $\mathbb{R}$-places, one $\mathbb{C}$-place

Prop 2 $\forall g \in \Gamma_n$, $\mathrm{tr}(g)$ is an alg. integer

Pf α, β generate ring of traces, $\alpha = 2\cos\frac{\pi}{n}$ and $\beta^2 - (1+\alpha^2)\beta + 2\alpha^2 - 1 = 0$.

Cor 2.1 (K, n) is arithmetic for $n = 4, 6, \infty$

Pf $k_n^{(2)}$ has no $\mathbb{R}$-places, so (iii) holds.

Cor 3.1 (of Lemma 1) Let $\Gamma \leq SL(2, \mathbb{C})$ with $\gamma_0 = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$ $\lambda^2 \neq 1$, $\gamma_1 = \begin{bmatrix} a & 1 \\ c & d \end{bmatrix}$ $c \neq 0$, $k =$ tr field of Γ, $[k : \mathbb{Q}] < \infty$, $\lambda \notin k$. If $\varphi : k \hookrightarrow \mathbb{R}$ s.t. $\varphi(a)^2 < 4$, TFAE:

(a) $\varphi(c) < 0$

(b) $A(\Gamma) \otimes_{\varphi(k)} \mathbb{R}$ is the quaternions $\qquad (A(\Gamma) := k[\Gamma] \leq M(2, \mathbb{C}))$

(c) $\varphi(\{\mathrm{tr}\, g : g \in \Gamma\})$ is bounded

For $n = 5, 8, 12$, $\quad \gamma_0 = A^2 = \begin{bmatrix} \lambda^2 & \\ & \lambda^{-2} \end{bmatrix} \quad \lambda^2 = e^{2\pi i/n}$

$$\gamma_1 = \begin{bmatrix} * & 1 \\ c & * \end{bmatrix} \quad c = \alpha^2(\mu(\alpha - \mu) - 1) \in k_n^{(2)}$$

They laboriously show $\varphi(c) < 0$ for every $k_n^{(2)} \hookrightarrow \mathbb{R}$.

This completes proof of thm 3.