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GRASP
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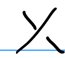
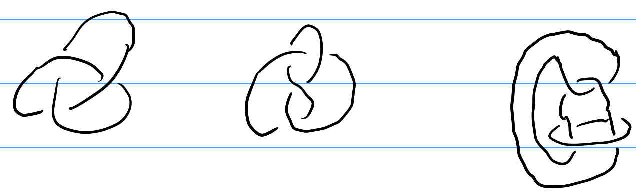
Kyle Miller

The Jones Polynomial

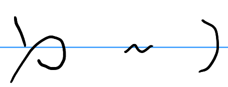
* Requisite knot theory

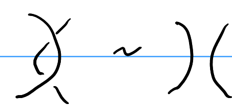
def A <u>link</u> L is a smooth/PL 1-dim'l closed submfld of S^3, up to <u>(ambient) isotopy</u> of S^3 — a smooth map $f:[0,1]\times S^3 \to S^3$ such that $f_t : S^3 \to S^3$ is a diffeomorphism for each t, with $f_0 = id$ and $f_1(L) = L'$.
A <u>knot</u> is a 1-component link.

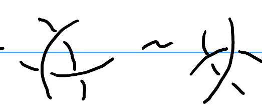
def A <u>knot/link diagram</u> D is a 4-regular graph embedded in S^2 whose vertices are marked like , representing a generic projection of a link.

ex 

Reidemeister moves:

RI) 

RII) 

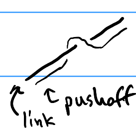
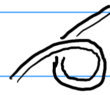
RIII 

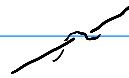
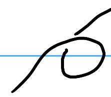
knots/links up to isotopy $\longleftrightarrow$ diagrams up to RI, RII, RIII

def A <u>framed knot/link</u> is a link along with a trivialization of its normal bundle, up to a suitable notion of isotopy

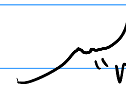
Only need a section, and represent as embedding of $L\times[0,1]$, with $L\times\{0\}$ the link and $L\times\{1\}$ the pushoff.

(2)

Diagram :  ~ 

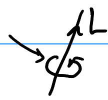
or:  ~  "blackboard framing"

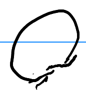
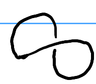
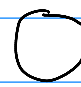
framed knots/links up to isotopy ⟷ diagrams up to RII, RIII, and

RI')  ~   "regular isotopy"

Framing is a $\mathbb{Z}^{\#\text{components}}$ – torsor.

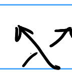
<u>Thm</u> There is a distinguished 0-framing of every component of an <u>oriented</u> link.

<u>Pf</u> Let Σ be a Seifert surface of L ($\partial\Sigma = L$, with induced orientation).
This is Poincaré dual to $\alpha \in H^1(S^3 - L)$ such that $\alpha(\mu) = 1$
with μ , for all such μ. Take a regular neighborhood of L in Σ □

<u>ex</u>  ~  has framing +1  +1 better notation

(b.b. framing) (with respect to 0-framing)

<u>def</u> Let D be an oriented diagram, and L the corresponding framed oriented link, from the blackboard framing.
The <u>writhe</u> $w(D) = \sum$ framings of $L \in \mathbb{Z}$.

<u>thm</u>  ↦ +1  ↦ −1 to compute $w(D)$

<u>ex</u> $w($  $) = 3$

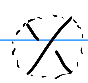
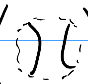
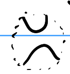
<u>thm</u> For knots, writhe is invariant under ori. reversal.

* Kauffman bracket (Kauffman 1987)

$\langle \text{diagram} \rangle \in \mathbb{Z}[A^{\pm 1}]$, $A = t^{-1/4}$

(i) $\langle \bigcirc \rangle = \delta := -A^2 - A^{-2}$ (traditionally 1)

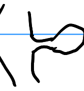
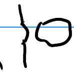
(ii) $\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$

(iii) $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

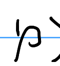
A-state, B-state

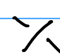
each diagram differs only inside 

so $\langle \text{diagram} \rangle = \sum_{\text{states}} A^{\#(A\text{-states}) - \#(B\text{-states})} \delta^{\#\text{loops}}$ (2^n terms) state sum

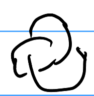
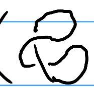
ex $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= \left(A + A^{-1}(-A^2 - A^{-2})\right) \langle \,|\, \rangle = -A^{-3} \langle \,)\, \rangle$

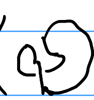
sim. $\langle$  $\rangle = -A^3 \langle \,)\, \rangle$

Can use  $\leftrightarrow$ 

$A \leftrightarrow A^{-1}$

ex $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

second term $= A^{-6}\delta$

first bracket $= A \langle$  $\rangle + A^{-1} \langle$  $\rangle = (-A^4 - A^{-4})\delta$

(with $-A^3\delta$ and $-A^{-3}\delta$ respectively)

$= (-A^5 - A^{-3} + A^{-7})\delta$

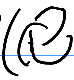
def The <u>Jones polynomial</u> of an oriented link L with an oriented diagram D is

$$V_L(t) = (-A^{-3})^{w(D)} \langle D \rangle \delta^{-1} \quad \text{with } A = t^{-1/4}$$

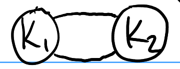
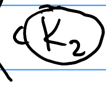
$(-A^{-3})^{w(D)}$: same as adding twists to zero-out framings

($L \neq \emptyset$ for this normalization)

④

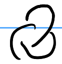
ex $V($  $)(t) = (-A^{-3})^3(-A^5 - A^{-3} + A^{-7})$

$= t + t^3 - t^4$

- $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$, and for knots $\in \mathbb{Z}[t^{\pm 1}]$
- $V(L_1 \amalg L_2) = \delta^{-1} V(L_1)\, V(L_2)$
- $V($  $) = V($  $)\, V($  $)$

  connect sum
- $V_{\bar{L}}(t) = V_L(t^{-1})$

  ↑ mirror image.  in diagrams

  ex  $\not\simeq$  (known to Dehn 1914)
- $\deg V_L(t) \leq$ crossing number $= \min_{D \text{ of } L}$ #crossings ... used to resolve a Tait conj. ...