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Nov. 2, 2017
Topics in Topology — Knot Theory
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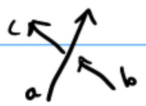
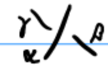
Racks and Quandles

Fox n-colorings (1956)

Given a link L, an n-coloring is a homomorphism $\varphi: \pi_1(S^3 - L) \to D_n = \langle \sigma, \tau \mid \sigma^n, \tau^2, \sigma\tau\sigma\tau \rangle$ such that meridian generators are sent to $\langle\sigma\rangle\tau$ (flips). The Wirtinger relations give



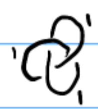
$$ac = ba \rightsquigarrow (\sigma^\alpha\tau)(\sigma^\gamma\tau) = (\sigma^\beta\tau)(\sigma^\alpha\tau)$$
$$\rightsquigarrow \sigma^{\alpha-\gamma}\tau = \sigma^{\beta-\alpha}\tau$$
$$\rightsquigarrow \gamma \equiv 2\alpha - \beta \pmod{n}$$

Represent such a φ by labeling diagram arcs like  (ori. doesn't matter)

The set $\mathrm{col}_n(L)$ of n-colorings under arc-wise addition is an abelian group, with the all-0 coloring the unit.

ex For $n=3$, this is $\alpha+\beta+\gamma \equiv 0 \pmod 3$. Only solns are all same/different.

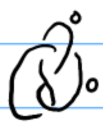
and



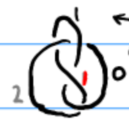
span $\mathrm{col}_3(3_1)$

↑ spans trivial colorings

But  has no non-trivial colorings.



→ all 0



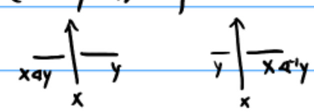
← WLOG by subtracting trivs and using $\mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$

Hence 3_1 and 4_1 are not equivalent.

Quandles (David Joyce, 1982) (also Takasaki '42, Conway-Wraith '50s, Matveev '82, Fenn-Rourke '92)

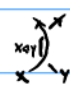
An algebraic structure satisfying "Reidemeister moves."

Let X be an object with a binary operator $\triangleleft$ ($x \triangleleft y$ is "y under x")

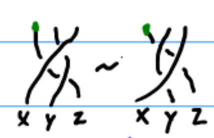


such that:

RI) For all $x \in X$, $x \triangleleft x = x$ 

RII) For all $x \in X$, $(y \mapsto x \triangleleft y) \in \mathrm{Aut}\, X$ 

RIII) For all $x,y,z \in X$, $x \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft (x \triangleleft z)$



Then $(X, \triangleleft)$ is a quandle. Without RI, is a rack.

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[**Lemma** $(x \triangleleft x) \triangleleft y = (x \triangleleft x) \triangleleft (x \triangleleft (x \triangleleft^{-1} y)) = x \triangleleft (x \triangleleft (x \triangleleft^{-1} y)) = x \triangleleft y$.

Cor $x \triangleleft x = y \triangleleft y =: z \Rightarrow$ 1) $z \triangleleft x = (x \triangleleft x) \triangleleft x = x \triangleleft x = z$ 2) $z \triangleleft y = \cdots = z$ $\Rightarrow$ $x = z \triangleleft^{-1} z = y$.

Cor $\pi(x) = x \triangleleft x$ is invertible, so racks are RI'-invariant

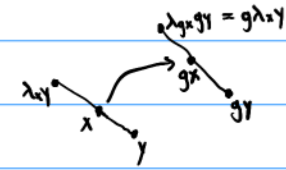
$\sim$) $\quad (x \triangleleft x) \triangleleft^{-1} (x \triangleleft x)$]

Exercise Racks satisfy RI') $\sim$). That is, $x \triangleleft x = y \triangleleft y \Rightarrow x = y$. (and $\rho(x) := x \triangleleft x$ is bijective)

ex $\mathbb{Z}/n\mathbb{Z}$ with $\alpha \triangleleft \beta := 2\alpha - \beta$ is the dihedral quandle $\mathcal{D}_n$.

main ex Let $G \curvearrowright X$ be a group action, and $\lambda : X \to G$ s.t. $g\lambda_x = \lambda_{gx} g$ $\forall g \in G$ and $x \in X$.

- $x \triangleleft y := \lambda_x y$ defines a rack
- If $\forall x \in X$, $\lambda_x x = x$, then is a quandle.



ex $c : G \to \operatorname{Aut} G$ by $c_g(h) = ghg^{-1}$ gives conj G.

$$\lambda_{gx} = g\lambda_x g^{-1} = c_g(\lambda_x) \Rightarrow \lambda_{\lambda_y x} = c_{\lambda_y}(\lambda_x) \Rightarrow \lambda(\underbrace{y \triangleleft x}_{\text{for } X}) = \lambda(y) \underbrace{\triangleleft}_{\text{for conj} G} \lambda(x)$$

So a rack comes with a rack hom. $X \to \operatorname{conj} \operatorname{Aut} X$.

ex If $\mathfrak{g}$ Lie alg. of Lie gp. G, $\exp : \mathfrak{g} \to G$ and $\operatorname{Ad} : G \to \operatorname{Aut} \mathfrak{g}$ makes $\mathfrak{g}$ a rack.

$$\operatorname{Ad}_{\exp(x)} x = \exp(\operatorname{ad}_x) x = x, \text{ so quandle.}$$

ex $X = S^n$ and $G = O(n+1)$. $\lambda : X \to G$ by $\lambda_x y = \operatorname{refl}_x y = 2\langle x, y\rangle x - y$

def If X a rack, $\operatorname{Adconj} X = \langle x \in X \mid \forall x, y \in X,\ xyx^{-1} = x \triangleleft y \rangle$ has univ. prop.:

$\forall$ gp G with $X \to \operatorname{conj} G$ a hom.,

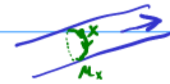
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \operatorname{Adconj} X \arrow[d, dashed, "\exists! \text{ group hom.}"] \\ & G \end{tikzcd}$$

Fundamental quandle of a link

Let L be oriented link, $* \in S^3 - L$ be basepoint,

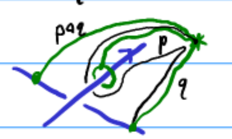
$Q(L)$ = homotopy classes of paths $*$ to $\partial N(L)$ (endpoint may slide)

$\pi_1(S^3 - L, *) \curvearrowright Q(L)$ by concatenation $(g \cdot p = gp)$



Up to isotopy, each $x \in \partial N(L)$ bounds unique meridian loop μ_x in $\partial N(L)$.

$$\lambda : Q(L) \longrightarrow \pi_1(S^3 - L, *)$$
$$p \longmapsto p\, \overline{\mu_{p(1)}}\, \overline{p}$$



Quandle:
- $g\lambda_p = g\, p\, \overline{\mu_{p(1)}}\, \overline{p} = g\, p\, \overline{\mu_{gp(1)}}\, \overline{gp}\, g = \lambda_{gp}\, g$
- $\lambda_p p = p\, \overline{\mu_{p(1)}} \sim p$

$Q(L)$ is fundamental quandle. The fund. rack for framed link analogous — framing instead of $\partial N(L)$.

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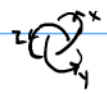
$\mathrm{Hom}(Q(L), X)$ is X-colorings.

$Q(L)$ can be presented by arcs in oriented diagram mod Wirt.-like relations.

ex $K=$  $Q(K) = \langle x,y,z \mid z = x \triangleleft y,\ y = z \triangleleft x,\ x = y \triangleleft z \rangle$
$\quad = \langle x, y \mid y = (x\triangleleft y)\triangleleft x,\ x = y \triangleleft (x \triangleleft y)\rangle$

$L =$  $Q(L) = \langle x, y \mid x = y \triangleleft x,\ y = x \triangleleft y\rangle$

prev ex $Q(K) \longrightarrow \mathcal{D}_3$ q.hom.
$x \longmapsto 0$
$y \longmapsto 2$

Thm $\mathrm{Adconj}\ Q(L) \cong \pi_1(S^3 - L)$.

Got here in 40 min.

Thm $Q(K)$ is a complete knot invariant.

Pf Fix $p \in Q(K)$. Let $G = \pi_1(S^3 - L)$ and $H = \mathrm{Stab}_G\, p$.

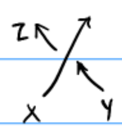
- For $g \in \pi_1(\partial N(K), p(1))$, $(p g \bar{p}) p \sim p$; hence $p\pi_1(\partial N(K), p(1))\bar{p} \subset H$.
- For $g \in H$, since $gp \sim p$, let h be path from $gp(1)$ to $p(1)$ over homotopy.

so $g = p\bar{h}\bar{p} \in p\pi_1(\partial N(K), p(1))\bar{p}$

Thus $H = p\pi_1(\partial N(K), p(1))\bar{p}$. Waldhausen '68 : peripheral system is complete invt. □

Let K be a knot, $G = \pi_1(S^3 - K)$, $H \subset G$ periph. subgp, and $\mu \in H$ a meridian.
G/H with $xH \triangleleft yH = x\mu^{-1}x^{-1}yH$ is isomorphic to $Q(K)$

Affine / Alexander quandles



$yx = xz \xrightarrow[\text{Fox deriv. for } H_1]{} y + tx = x + tz \longrightarrow z = (1-t^{-1})x + t^{-1}y$

$x \triangleleft y := (1-t^{-1})x + t^{-1}y$ gives an affine quandle
($x, y \in A$ an R-mod, $t \in \mathrm{Aut}\, A$)

ex $A = \mathbb{Z}/n\mathbb{Z}$, $R = \mathbb{Z}$, $t = -1$ gives dihedral quandle

Q: When is there hom. $Q(K) \longrightarrow A$ with non-triv. image ?

Most-general A is $H_1((S^3-K)_\infty)$ over $\mathbb{Z}[t^{\pm 1}]$ as an affine quandle.
$\cong \mathbb{Z}[t^{\pm 1}]/(\Delta_K(t))$

ex $t = -1$, is $\mathbb{Z}/\Delta_K(-1)\mathbb{Z}$. Have non-triv. Fox n-coloring if $n \mid \Delta_K(-1)$.
$\Delta_K(-1) \in 2\mathbb{Z}+1$. 10_{124} is first non-triv. knot with $|\Delta_K(-1)| = 1$.

$$\begin{tikzcd}
Q(K) \arrow[r] \arrow[d, "f"] & \pi_1(S^3 - K) \arrow[d, dashed] \\
\mathbb{C}[\alpha] \arrow[r] & \mathrm{AdConj}\,\mathbb{C}[\alpha] \cong \mathrm{Aff}\,\mathbb{C}
\end{tikzcd}$$

affine representation with meridians: $g \mapsto \begin{bmatrix} \alpha & f(g) \\ 0 & 1 \end{bmatrix}$

$\uparrow$ root of $\Delta_K(t)$

"twisted 1-cocycle"

- Quandles also have applications to sfc knots in S^4
- Cohomology more sophisticated invt. than $\mathrm{Hom}(Q(K), X)$.

$\}$ Carter