Spatial graph invariants

(for 3-mfld seminar)

9/19/2017 $\boxed{\text{Part II}}$

A spatial graph is an embedding of a finite 1-dim CW cplx in S^3, equivalence up to isotopy. (For simplicity, think PL.)

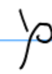
ex  vs  (theta graphs)

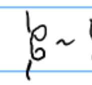
A flat vertex graph is when vertices are oriented disks.

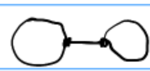
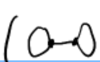
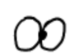
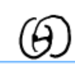
ex  vs 

A ribbon graph is when vertices and edges are disks. (2-dim mfld w/ ∂)

ex  vs  vs 

Can represent orientable ribbon graphs by relying on blackboard framing (top-side-up)  $\leftrightarrow$  and  $\leftrightarrow$  ~ 

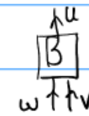
F.v. graph diagrams up to regular isotopy (~~RI.~~ $\rightsquigarrow$ RI'. ) is isotopy of corresponding ribbon graph.

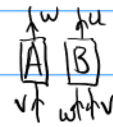
ex Failure of π_1.  vs . Not isotopic, but complements of regular nbhds are homeomorphic. ( $\leftrightarrow$  $\leftrightarrow$ )
Yamada polynomial can detect difference. (A Reshetikhin-Turaev invt.)

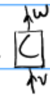
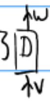
Story Some algebras have a graphical notation resembling braids/links/graphs respecting Reidemeister-like moves. So, take a graph diagram and pretend it is in the algebra!

Plan Graphical notations, invts of 2D immersions, invts of spatial graphs.

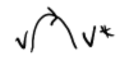
* Penrose graphical notation '71

Map $A: V \to W$  $B: W \otimes V \to U$ 

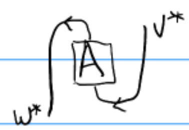
Tensoring by juxtaposition:  $= A \otimes B : V \otimes W \otimes V \to W \otimes U$

Linearity by $+$: 2  $+ 3$ 

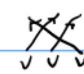
Contraction/composition with arcs:  $: V \otimes V \to U$

Identity map $\uparrow v$ and dual pairing  $= \sum_i v^i \otimes v_i$ (assume f.dim.)

also "co-dual" ("Casimir")  $= \sum_i v_i \otimes v^i$

<u>ex</u> dual of A is  $: W^* \to V^*$.  $\in V^* \otimes W$ is "matrix"

Vect has transpositions  $: V \otimes W \to W \otimes V$

Repr. $\mathbb{C}[S^n] \to \operatorname{End}(V^{\otimes n})$ by $\sigma \mapsto \sigma_*$. <u>Ex</u> $(1\,2\,3) \mapsto$ .

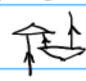
<u>Trace</u> Let $T: V \to V$.  $= \left(\sum_i v_i \otimes v^i\right) \circ \mathrm{id}_{V^*} \otimes T \circ \left(\sum_j v^j \otimes v_j\right)$

$= \sum_{i,j} v_i(v^j) \cdot v^i(T(v_j)) = \sum_i v^i(T(v_i)) = \operatorname{tr} T.$

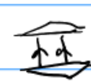
<u>ex</u>  $= \operatorname{tr} \mathrm{id}_V = \dim V.$

<u>Metrics</u> Suppose  $\in V^* \otimes V^*$ is a non-degenerate bilinear form.

(ie.,  $: V \to V^*$ is an isomorphism)

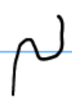
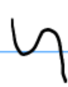
Let  be inverse :  $= \uparrow$.

1) If <u>symmetric</u>,  $=$  $=$  $=$  $= \dim V$

2) If <u>antisymmetric</u>,  $= -$  $= -\dim V$

3) In either case,  $=$  $= \uparrow$.

Convention: fix sym/alt form & drop arrows. So:

 $=$  $=$  and  $= \dim V \cdot \begin{cases} 1 & \text{if sym} \\ -1 & \text{if antisym} \end{cases}$

$= 1$ if sym, $= -1$ if antisym, so $= 1$ for both.

ex antisym gives $\mathbb{Z}/2\mathbb{Z}$ winding # for closed curve.

*** so(3) invariant of 3-valent (cubic) fat graphs.** (Penrose '71. "Apps of neg.-dim tensors")

Take $V = \mathbb{C}^3$, $\cap$ = dot product, $= i \det$

$(3|V| = 2|E|$, so will give $\mathbb{R}$ elt.)

$=$, etc. ($= i \cdot$ cross product)

Lemma $=$ $-$

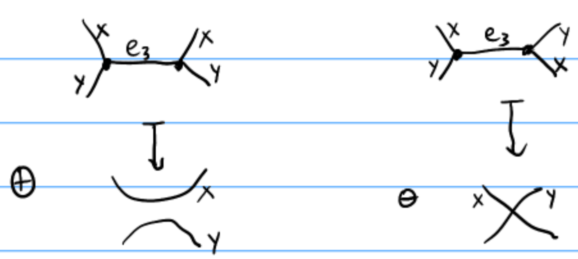
Pf Let a,b,c be some permutation of std. basis. Non-zero evaluations must be of form $=$ or $= -$

ex $=$ $-$ $= 3^2 - 3 = 6$

$= -6$ since det alternating ($= -$)

Thm (Penrose '71, pf by Kauffman) Image of graph Γ = # edge 3-colorings of Γ if Γ is planar.

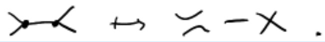
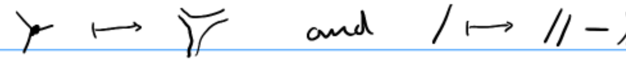
Pf Let e_1, e_2, e_3 be std basis. Value of im of Γ is sum over all assignments of e_1, e_2, e_3 to edges. Only care about non-zero assignments, which is when distinct vectors around each vtx. Consider e_3 edges and simplify:

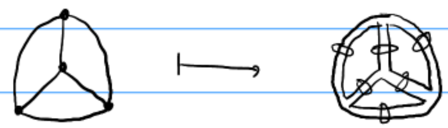


Result: a collection of simple closed curves labeled e_1 & e_2.

$\mathbb{Z}/2\mathbb{Z}$ intersection # $= 0$ $\Rightarrow$ even # of $\ominus$'s, so this assignment is a positive summand, contributes $+1$. $\square$

"State sum"  so can do

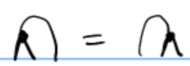
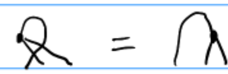
Recall  . Then can compute from orig. graph by  and $=: \#$

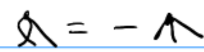
ex  Can expand #'s to get 2^6 terms.

$$= \sum_{\text{expansions}} (-1)^{\# X\text{'s}}\, 3^{\#\text{circles}}$$

* Lie alg. generalization

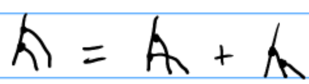
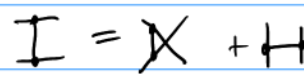
Let $\mathfrak{g}$ be semisimple Lie alg. $\cap$ its Killing form, $\curlywedge$ its Lie bracket and $\cup$ its Casimir.

(i)  (ii) 

hence  is rotationally invariant. Alt:  (D_3 sign representation)

Get map $\mathbb{C}[\text{flat vtx graphs}] \to \mathbb{C}$

Prev: $\mathfrak{g} = so(3) \cong sl(2)$.

Jacobi :  , or the famous 

ex  (So $K_{3,3}$ gives 0 universally)

More universal relations in: Chmutov, et al. "The algebra of 3-graphs."
(Discusses inj. map to $\mathbb{Z}$-homology S^3's, from Le '97)

* Penrose polynomials

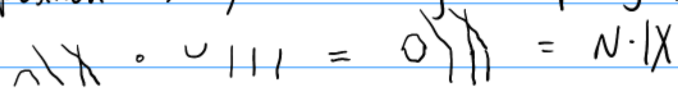
$W_{so(N)}(\Gamma)$, $W_{sl(N)}(\Gamma)$ are above constructions, functions of N.
(Γ an immersed f.v. graph)

def The Brauer category $Br(N)$ with $N \in \mathbb{C}$ has

objects $\underline{n} = \bullet \bullet \cdots \bullet$ (n points) $n \in \mathbb{N}$

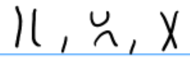
morphisms $Br_{m,n}(N)$ from $\underline{m}$ to $\underline{n}$ of abstract tensor diagrams connecting pts involving permutations, an "inner product" $\cap$, and corresponding $\cup$.

Composition is by connecting corresponding pts on boundary, with $\bigcirc = N$.

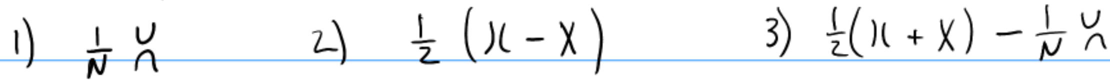
ex 

$\in Br_{5,3}(N)$ $\quad \in Br_{3,5}(N)$ $\quad \in Br_{3,3}(N)$

$Br_n(N) := Br_{n,n}(N)$ is Brauer algebra. $Br_n(N) \twoheadrightarrow \mathrm{End}_{so(V)}(V^{\otimes n})$ is from $so(V)$ Schur-Weyl duality (Brauer '37)

Basis of $Br_2(N)$: 

Primitive idempotents of $Br_2(N)$: (semisimple)

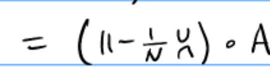


Consider $\mathfrak{g} < \mathfrak{gl}(N)$, $V = \mathbb{C}^N$.

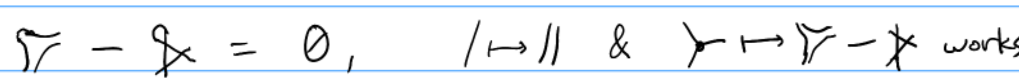
$A \in \mathfrak{g}$ can be represented as a matrix  $\in V \otimes V$

$A, B \in \mathfrak{g}$, $AB =$ 

$\mathrm{tr}(AB) =$ 

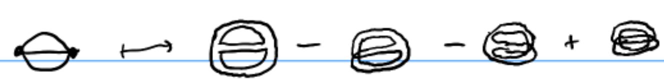
For $\mathfrak{g} = \mathfrak{sl}(N)$, Killing form proportional to trace form, but restricted to traceless matrices. Projector : $A \mapsto A - \frac{\mathrm{tr} A}{N} I$ 

Hence  calculates $W_{\mathfrak{sl}(N)}(\Gamma)$

Simplification Since 

(this is just $\mathrm{tr}[A,B] = 0$) (If not degenerate case of $\bigcirc$)

$= \dim \mathfrak{sl}(N)$

ex  $= N^3 - N - N + N^3$

$= 2N(N^2 - 1) = 2N \dim \mathfrak{sl}(N)$

For $\mathfrak{g} = so(N)$, Killing form same, and trace form prop. to Killing form.

Projector $gl(N) \to so(N)$ is $A \mapsto \frac{1}{2}(A - A^T) = \frac{1}{2}\Big($  $-$  $\Big) = \frac{1}{2}(\,|| - X\,) \circ A$.

Let  $= || - X$.  $= -$  so  $= -$ 

So, up to scaling, can use $| \longmapsto$  and  $\mapsto$  for $W_{so(N)}(\Gamma)$.

$\#$ edge 3-colorings $= W_{so(3)}(\Gamma) \simeq 2^{v-e}\, W_{sl(2)}(\Gamma)$.

(two different-seeming ways to calculate!)

Part II

9/26/17 Last time: tensor diagrams, Penrose polys

A cellular/combinatorial embedding of a graph Γ into a surface Σ is one where Γ is $\Sigma^{(1)}$ (i.e., no genus holes in faces; Σ obtained by gluing disks).

ex  not  (dual graphs are w.r.t. such embeddings)

A planar graph is a graph with a cellular emb. to S^2.

Same as giving abstract ribbon structure. Thicken Γ in sfc, or attach disks to ∂ of ribbon graph to get sfc.

Recall $W_{sl(N)}(\Gamma)$ for cubic f.v. graph Γ:

($\| - \frac{1}{N} \cup\cap$ generally)

in $Br(N)$, where $O = N$, $\rho = 1$, etc.

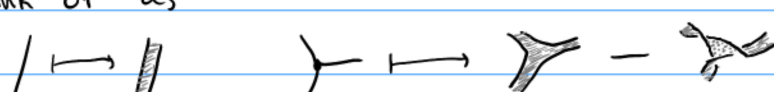
Thm (Bar-Natan '97) Coefficient of N^f in $W_{sl(N)}(\Gamma)$ is a signed count of cellular embeddings of f.v. graph Γ into oriented genus-$(1 - \frac{f}{2} + \frac{v}{4})$ sfcs.

Sign of emb. $= \prod_v \varepsilon_v$ where $\varepsilon_v = \begin{cases} +1 & \text{if } v \text{ matches ori. of } \Sigma \\ -1 & \text{if } v \text{ reversed ori.} \end{cases}$

In Θ example,

f	g	signed count
3	0	2
1	1	-2

Pf Think of as



for ribbon structure of embeddings (-1 for local accounting of sign)

An individual term gives $\pm N^f$ where $f = \#$ bdry S^1's $= \#$ attached D^2's.

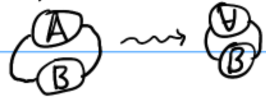
$$v - e + f = 2 - 2g \qquad 3v = 2e \text{ (cubic)}$$

so $g = 1 - \frac{f}{2} + \frac{v}{4}$. $\square$

(Can look at poly $x^{1+\frac{v}{4}} W_{sl(x^{-1/2})}(\Gamma) =: \sigma(\Gamma)$ so x^g term is for genus-g cell. embs.)

Thm Let $W^{top}_{sl(N)}(\Gamma)$ = coeff of $N^{2+\frac{v}{2}}$.

$|W^{top}_{sl(N)}(\Gamma)|$ = # planar embeddings of Γ.

Pf Follows from Whitney '33: All planar embs of cubic Γ are related by moves . A,B contain even # verts, each, so sign is constant. $\square$

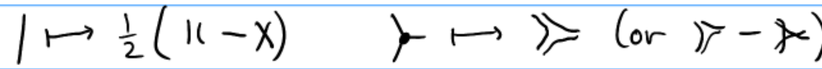
Restatement of 4-color thm (Bar-Natan '97) (Using Tait colorings)

Thm $W^{top}_{sl(N)}(\Gamma) \neq 0 \Rightarrow W_{sl(2)}(\Gamma) \neq 0$.

Pf (?)

(Aside:  counts all cell. embs.)

* **Coincidence**

Recall $W_{so(N)}(\Gamma)$ via 

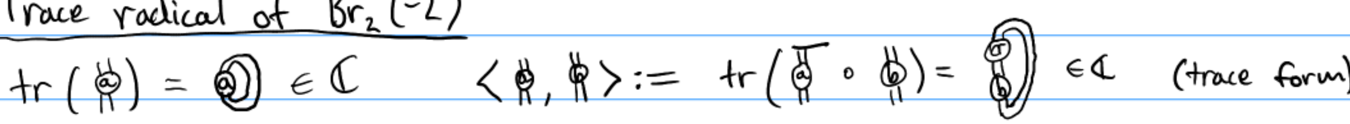
Thm (Penrose '71) $W_{so(-2)}(\Gamma) \doteq W_{so(3)}(\Gamma)$

up to $a^v b^e$ for fixed a, b.

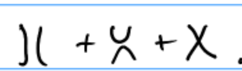
will prove.

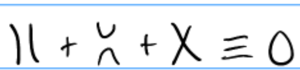
(**Thm** (Szegedy '02) $\doteq W_{so(4)}(\Gamma)$, too.)

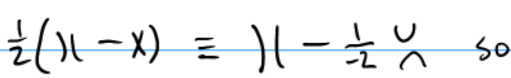
* **Trace radical of $Br_2(-2)$**



Degenerate elements of a trace form (the trace radical) give relations for closed graphs (since they are all trace of graph with cut-open edge)

Trace radical of $Br_2(-2)$ gen by . Working in quotient,

 is binor identity.

 so $W_{so(-2)}(\Gamma) = W_{sl(-2)}(\Gamma)$.

Later, $\doteq W_{sl(2)}(\Gamma)$.

* $U_q(sl(2))$

Warmup: $V = \mathbb{C}^2$ defining repr. of $sl(2)$

1) $\cap = \det$ is invt. 2-form. $\bigcirc = -2$

2) $X = (1\ 2)_*$ is $sl(2)$-int.

def $\underbrace{\quad}_{n} = \sum_{\sigma \in S_n} \sigma_*$ is symmetrizer

$\underbrace{\quad}_{n} = \sum_{\sigma \in S_n} (-1)^{\sigma} \sigma_*$ is antisymmetrizer

$= 0$ since $\dim V = 2$ and $\dim \operatorname{im}(\frac{1}{3!}\ ) = \dim \Lambda^3 V = 0$

so $0 = \ = \ - \ - \ - \ + \ + \ = 2(\,|| + \ - X)$

hence $X = || + \ $

3) Schur-Weyl duality $\Rightarrow \mathbb{C}[S^n] \twoheadrightarrow \underbrace{\operatorname{End}_{sl(2)}(V^{\otimes n})}$

TL_n^{-2} Temperley-Lieb algebra.

4) Irred. reprs are $V_n := \operatorname{Sym}^n V = \operatorname{im}(\frac{1}{n!}\ \underbrace{\quad}_{n})$

$V_2 \cong$ adjoint repr. Projector is $:= \frac{1}{2} = \frac{1}{2}(|| + X) = || + \frac{1}{2}$

5) Clebsch-Gordon $\Rightarrow V_2 \otimes V_2 \cong V_4 \oplus V_2 \oplus V_0$, so $\dim \operatorname{Hom}_{sl(2)}(V_2 \otimes V_2, V_2) = 1$.

$\Rightarrow$ is only homomorphism up to scaling. (so Lie bracket of $sl(2)$)

$\Downarrow$

$| \mapsto$ $\quad$ $\mapsto$ $\quad$ $X \mapsto$ $\quad$ gives $W_{sl(2)}(\Gamma)$ (up to renorm)

(This is $W_{sl(-2)}(\Gamma)$ if you squint and don't move strands around...)

Can extend to $X \mapsto$, etc.

ex Can distinguish from (f.v. graphs)

Deformation to $U_q(sl(2))$: ($q=1$ is $sl(2)$) V a 2-D irred. repr.

1) $\bigcirc = -[2]_q = -q - q^{-1}$

2) $= q^{1/2})(+ q^{-1/2}$ (and rotation), $= ||$, $=$, $= -q^{3/2}\, |$

3) $\mathbb{C}[B_n] \to \operatorname{End}_{U_q(sl(2))}(V^{\otimes n})$

4) finite-dim. irred. reprs are $V_n := \operatorname{im}$. Jones-Wenzl projector $p^{(n)}$

$= || + \frac{1}{[2]_q}$

5) Clebsh-Gordon $\Rightarrow$ unique-up-to-scaling 3-form on V_2.

ex Kauffman bracket of a link diagram ('87)

$$= \frac{1}{-[2]_q} \cdot \left(\text{link as } U_q(sl(2))\text{-inv't fn (color by } V_1)\right)\Big|_{q=A^2}$$

Jones poly of link = add ⟲'s to make 0 writhe then $K.B.\big|_{A=t^{-1/4}}$

* Yamada polynomial

2^{nd}-colored Jones poly for graphs, i.e. Reshetikhin-Turaev inv't for $U_q sl(2)$ & V_2-coloring

Let $V_2(\Gamma, q) = \left(| \mapsto \#, \ Y \mapsto \ldots \right)$

$$Y_\Gamma(A) = (-[2]_q)^{e-v}\, V_2(\Gamma, q)\Big|_{q=A^{1/2}}$$ is a Laurent polynomial in A.

⟲ $= A^2\,|$ $\qquad$ $= -A$ $\qquad$ so Y is invariant of

- spatial graphs up to factor of $-A$
- f.v. graphs up to factor of A^2
- ribbon graphs

ex $Y(\text{O—O}) = \emptyset$

$Y(\ldots) = A^{-5} + A^{-4} + \cdots - A^4 \neq \emptyset$

silly ex: Y cannot distinguish between ribbon graphs ⊖ and ... (since they are isotopic!)

Defining properties

- $Y(\bigcirc) = A + 1 + A^{-1}$
- $Y(\times) = A\,Y(\,)(\,) + A^{-1}Y(\asymp) - Y(\ldots)$
- $Y(\ldots) = Y(\ldots) - Y(\ldots)$ if e not a loop. (contraction-deletion)
- $Y(\Gamma_1 \amalg \Gamma_2) = Y(\Gamma_1)\,Y(\Gamma_2)$
- $Y(\Gamma) = \emptyset$ if Γ has cut edge. Γ_1—Γ_2 $\left(\dim \mathrm{Hom}_{U_q sl(2)}(V_2, V_0) = \emptyset\right)$

(**Thm** (Jaeger '89 (?)) $Y_\Gamma(A) = F_\Gamma((A+A^{-1})^2)$ for Γ planar. $F_\Gamma(n)$ = # nonvanishing abelian flows, for ab. gp. of order n.)

*Relationship to Penrose

- $Y_\Gamma(1) \doteq W_{sl(2)}(\Gamma)$ (since $Y_\Gamma(1)$ is for $q^{1/2}=1$)
- $Y_\Gamma(1)$ is also for $q^{1/2}=-1$ case.

  in particular, $\times = -)(- \asymp$, the binor identity

  so $Y_\Gamma(1) \doteq W_{sl(-2)}(\Gamma)$

Hence $W_{so(-2)}(\Gamma) \doteq W_{so(3)}(\Gamma)$.

*Dubrovnik poly

2-var extension. (Kauffman "An invariant of regular isotopy")

Birman-Murakami-Wenzl algebra

BMW(a,z) is category, similar in constr. to Br. Same as Kauffman tangle alg. over $\mathbb{C}(a,z)$

$$\curlyvee = a\,\big| \qquad \curlyvee = a^{-1}\,\big| \qquad \times - \overline{\times} = z\big(\,)(- \asymp\big)$$

$$\Rightarrow \bigcirc = 1 + \frac{a-a^{-1}}{z} =: \delta \qquad \text{Let } z = q - q^{-1}$$

Schur-Weyl relationship for 3-dim irrep of $U_q\,sl(2)$

$D_L(a,z)$ for links, divided by δ.

Basis for $BMW_2(a,z)$ is $)(, \asymp, \times$

Primitive idempotents:

(1) $e_1 = \delta^{-1} \asymp$

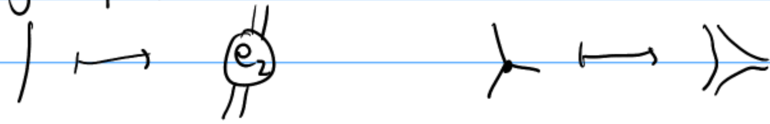
(2) $e_2 = \frac{1}{q+q^{-1}}\left(q^{-1}\,)(- \frac{1+aq^{-1}}{\delta a} \asymp + \times\right)$

(3) $e_3 = \frac{1}{q+q^{-1}}\left(q\,)(+ \frac{1-aq^{-1}}{\delta a} \asymp - \times\right)$

$q \mapsto -q^{-1}$ swaps $e_2 \leftrightarrow e_3$

non-triv. trace radical if $a = \pm 1$, $\underbrace{a = q^{-3}, \text{ or } a = -q^{3}}_{\text{Give Kauffman bracket.}}$

* Jaeger polynomial ('89)



Specializes to Y (up to some renormalization)

* Deformed Penrose polys

If $a = q^{N-1}$ with $N \in \mathbb{Z}$

if $q \to 1$, $\delta \to N$ and $z \to 0$

so get $Br(N)$

$$e_1 \longrightarrow \frac{1}{N} \substack{\cup \\ \cap}$$

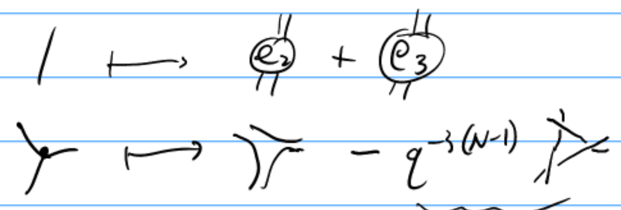
$$e_2 \longrightarrow \frac{1}{2}\,)(\; - \frac{1}{N} \; + \frac{1}{2} X$$

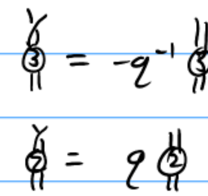
$$e_3 \longrightarrow \frac{1}{2}\,)(\; - \frac{1}{2} X$$

$$e_1 + e_2 \longrightarrow)(\; - \frac{1}{N} \substack{\cup \\ \cap}$$

$$=)(\; - \delta^{-1} \substack{\cup \\ \cap}$$

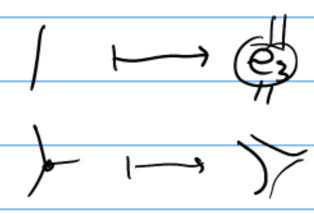
so •



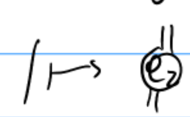


is a 2-var generalization of $W_{sl(N)}$

•



is a 2-var generalization of $W_{so(N)}$

(or 

with different limit)