Student 3-mfld seminar

2/8/19
KM

The loop thm & Dehn's lemma

Stalling's formulation: (1971)

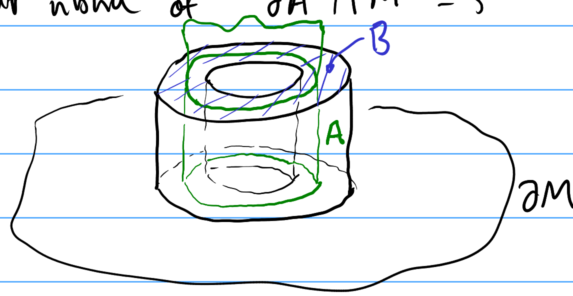
Thm (Loop theorem) Let M be a 3-mfld, $B \subset \partial M$ a compact surface, $N \triangleleft \ker(i_*: \pi_1(B) \to \pi_1(M))$ a proper normal subgroup. Then there is a proper embedding $f: (D^2, S^1) \to (M, B)$ such that $f|_{S^1}$ represents a free homotopy class outside N.

↳ a conjugacy class of $\pi_1(B)$

Stated by Dehn 1910 with incorrect proof:

Cor (Dehn's Lemma) Let M be a 3-mfld and let $f: (D^2, S^1) \to (M, \partial M)$ be a map that on a collar neighborhood A of S^1 is a proper embedding. Then there is a proper embedding $f': (D^2, S^1) \to (M, \partial M)$ s.t. $f'|_A = f|_A$.

Pf Let M' be from cutting out a regular nbhd of A, and let $B \subset \partial M'$ be a regular nbhd of $\partial A \cap M' \cong S^1$

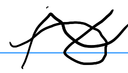


For $B \subset \partial M'$ and $N = 1 \triangleleft \pi_1(B)$, apply the Loop Thm to get a proper embedding $f': (D^2, S^1) \to (M', B)$ s.t. $f'|_{S^1}$ is essential in B. $f'|_{S^1}$ is isotopic to $\partial A \cap M'$ (possibly with reversed orientation) and this isotopy extends, so can assume $f'|_{S^1} = \partial A \cap M'$, WLOG with correct orientation. Thus can extend domain of f' to $D^2 \cup A$. ∎

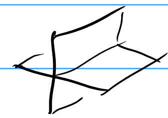
* Singularities

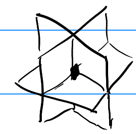
What are the failures in C^0-approximating arbitrary maps by immersions in general pos.?

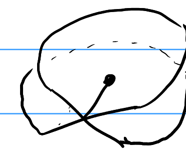
- 1-mfld $\longrightarrow$ 2-mfld

  Only isolated transverse double points. 

- 2-mfld $\longrightarrow$ 3-mfld

  double curves: 

  triple points: 

  And, if not already an immersion, branch points: 
  of order 2

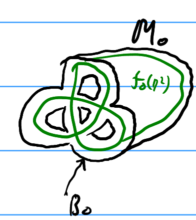
$*$ Simplicial approximation

For K, L compact simplicial complexes, $f: K \to L$ any map, then f is homotopic to a simplicial map with some subdivision of K. (With a fine enough subdiv., each simplex of K is taken into a star of a vtx of K. Extend linearly.)

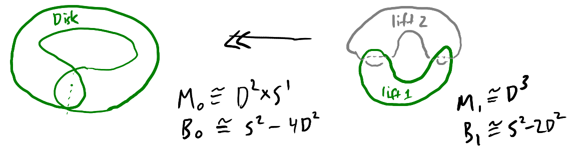
$*$ Tower construction (Papakyriakopoulos 1957)

Start with
- M a 3-mfld
- $B \subset \partial M$ compact sfc, $i: B \hookrightarrow M$
- $N \trianglelefteq \ker i_*$ proper
- a proper map $f: (D^2, S^1) \longrightarrow (M, B)$ s.t. $[f|_{S^1}] \notin N$

Let
- f_0 be simplicial approx of f
- M_0 be closed regular nbhd of $f_0(D^2)$
- $B_0 = B \cap M_0$
- $N_0 \trianglelefteq \pi_1(B_0)$, $N_0 = \pi_1(B_0 \hookrightarrow B)^{-1}(N)$



Note: $[f_0|_{S^1}] \notin N_0$



Suppose (M_0, B_0) has a connected double cover $p_0 : (M_0', B_0') \longrightarrow (M_0, B_0)$.

Can lift f_0 to f_1:

$$\begin{tikzcd} & (M_0', B_0') \arrow[d, "p_0"] \\ (D^2, S^1) \arrow[ur, dashed, "f_1"] \arrow[r, "f_0"'] & (M_0, B_0) \end{tikzcd}$$

Let

- M_1 be closed regular nbhd of $f_1(D^2)$
- $B_1 = B_0' \cap M_1$
- $N_1 = \pi_1(B_1 \hookrightarrow B_0' \xrightarrow{p_0} B_0)^{-1}(N_0)$

One gets a tower by inductively repeating this construction

$$\begin{tikzcd} & (M_i, B_i) \subset (M_{i-1}', B_{i-1}') \arrow[d, dashed, "p_{i-1}"] \\ & \vdots \quad (M_2', B_2') \arrow[d, "p_2"] \\ & (M_2, B_2) \subset (M_1', B_1') \arrow[d, "p_1"] \\ & (M_1, B_1) \subset (M_0', B_0') \arrow[d, "p_0"] \\ (D^2, S^1) \arrow[uuuur, "f_i"] \arrow[uur, "f_2"] \arrow[ur, "f_1"] \arrow[r, "f_0"] & (M_0, B_0) \subset (M, B) \end{tikzcd}$$

Lemma Every tower is finite.

Pf Let $\phi(f_i) = \underbrace{(\#\text{simplices in } D^2)}_{\text{constant for all } i} - \underbrace{(\#\text{simplices in } f_i(D^2))}_{\text{strictly increasing}}$.

since if not, $M_{i+1} \cong M_i \Rightarrow$ disconnected cover

Since $\phi(f_i) \geq 0$, there is a maximal i $\blacksquare$

Lemma If ∂M has a non-sphere component, then M admits a connected dbl. cover

Pf Recall: $H^1(M; \mathbb{Z}/2\mathbb{Z})$ parameterizes double covers $(\cong \mathrm{Hom}(H_1(M), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}))$

Suppose $H^1(M; \mathbb{Z}/2\mathbb{Z}) = 0$. Poincaré duality $\Rightarrow H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = 0$. UCT $\Rightarrow H_1(M; \mathbb{Z}/2\mathbb{Z}) = 0$.

LES has $H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z}/2\mathbb{Z}) \Rightarrow H_1(\partial M; \mathbb{Z}/2\mathbb{Z}) = 0$

Hence ∂M is all spheres. $\blacksquare$

Consider the last stage $(D^2, S^1) \xrightarrow{f_i} (M_i, B_i)$.

B_i is planar surface, so $\pi_1(B_i)$ is normally generated by ∂B_i loops.

$\Rightarrow \exists$ loop in ∂B_i not in N_i. This loop bounds a disk in ∂M_i (spheres)

So let $g_i : (D^2, S^1) \to (M_i, B_i)$ be a properly embedded disk (push disk slightly in)

with $[g_i(S^1)] \notin N_i$

We now induct down on i to prove Loop Thm:

Let $g_i' = p_{i-1} \circ g_i : (D^2, S^1) \longrightarrow (M_{i-1}, B_{i-1})$, perturb to general position.

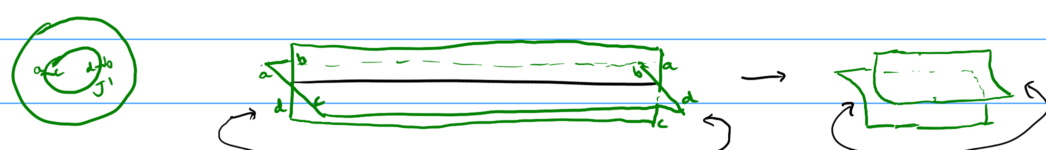
Singularities:

- double arcs and loops
- no triple points or branch pts since from double cover

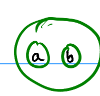
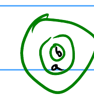
Now, modify g_i' to produce a proper embedding.

a) Suppose there is a double loop $J \subset g_i'(D^2)$, let J' = preimage of J

i) If $g_i' : J' \to J$ is connected double cover, can cut D^2 along J', twist by π, and reglue; g_i' perturbed eliminates J'



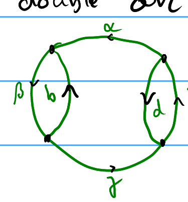
identify w/ ori.-reversing map

ii) If $J' \to J$ is disconnected, either  or . Suppose b innermost loop. (over all double loops)

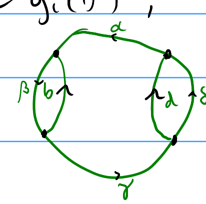
Cut out disk a and replace with disk b, perturb. Eliminates ≥ 2 loops.

Now no double loops

b) Suppose there is a double arc $J \subset g_i'(D^2)$, J' = preimage. Take so an arc in J' is innermost. (label b)



or



options:

i) glue lunes together   ii) replace d lune with b, perturb

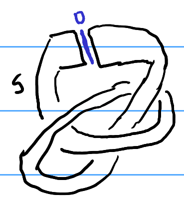
boundary: i) $\beta^{\pm}\delta$ ii) $\alpha\beta\gamma\beta^{\mp}$. $(\alpha\beta\gamma\beta^{\mp})(\beta^{\pm}\delta) = \alpha\beta\gamma\delta \notin N_{i-1}$, so one $\notin N_{i-1}$

Thus, get an embedding $g_{i-1} : (D^2, S^1) \to (M_{i-1}, B_{i-1})$.   When $i = 0$, g_0 is the emb. disk ∎

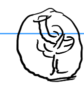
Kneser's lemma Let S be a $\overbrace{\text{2-sided}}^{\partial\nu(S)\text{ has two components}}$ sfc in a 3-mfld M. If S is not π_1-injective (i.e., $\ker(\pi_1(S) \to \pi_1(M)) \neq 1$) then there is a proper emb. $(D^2, S^1) \longrightarrow (M - \nu(S), \partial\nu(S))$ whose boundary is essential in $\partial\nu(S)$

Example Let $T^2 \subset S^3$ be an embedded torus. $\pi_1(T^2) \to \pi_1(S^3)$ not injective, so there is a properly embedded disk D with $\partial D \subset T^2$.

Compress T^2 along D, yielding a sphere S. Alexander's thm $\Rightarrow$ S bounds a ball on either side; let B be on side opposite D. "Uncompressing" B along D gives a $S^1 \times D^2$ that T^2 bounds.



Hence: T^2 is ∂ of $\nu(\text{knot})$.

[for $T^2 \subset \mathbb{R}^3$, $\Rightarrow$ this or a knot complement ]

Pf Let $\gamma: S^1 \to S$ represent an essential loop that is inessential in M — i.e., there is a nullhomotopy $f: (D^2, S^1) \longrightarrow (M, S)$ with $f|_{S^1} = \gamma$.

Since S is 2-sided, can push γ off S to one side — so assume now $f(S^1)$ does not intersect S.

Since S is embedded, can perturb f so $f^{-1}(S)$ is disjoint simple closed curves.

Take innermost $L \subset f^{-1}(S)$:

- If $f(L)$ inessential in S, there is a disk near S that the interior of L may be replaced by, removing an intersection
- If $f(L)$ essential in S, replace f by restriction to interior disk, removing all intersections

Thus, can assume $f(D^2) \cap S = \emptyset$, hence $\partial\nu(S) \longrightarrow M - \nu(S)$ is not π_1-injective. Apply the Loop Theorem. ∎

Freedman & Scharlemann 2017:

Thm Let M be a 3-mfld with collar nbhd $\partial M \times [0,\varepsilon]$,
$\gamma : S^1 \to \partial M$ an immersion in general position
whose inclusion in M is nullhomotopic. Then there is a (Morse) displacement
fn $\delta : S^1 \to (0,\varepsilon)$ such that $(\gamma \times \delta) \circ \Delta : S^1 \to \partial M \times [0,\varepsilon]$ is an
embedded loop bounding an embedded disk.