Student 3-mfld seminar

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$\boxed{\text{The Sphere theorem}}$

A sfc $\Sigma \subset M^3$ is <u>two-sided</u> if its normal bundle is trivial.
(i.e., $\nu(\Sigma) \cong \Sigma \times [-1,1]$ with $\Sigma \cong \Sigma \times \{0\}$.)
If M orientable: Σ two-sided $\iff$ Σ orientable

Papakyriakopoulos 1957, Epstein 1961

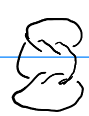
<u>Thm</u> (Sphere Theorem / Projective Plane Theorem) Let M^3 be compact, $N \trianglelefteq \pi_2(M)$ a $\pi_1(M)$-invariant proper subgroup. Then there is an embedded two-sided S^2 or $\mathbb{R}P^2$ Σ s.t. $[\Sigma] \notin N$. ($N = 1$ common.)

For non-compact M: take $f: S^2 \to M$ with $[f] \notin N$ and restrict to $\nu(f(S^2))$.
In general, can get a map in $f(S^2) \cup$ singular set.

<u>ex</u> Let $K \subset S^3$ be a knot (i.e., $K \cong S^1$). $S^3 - K$ is a <u>knot complement</u>.

1. $S^3 - K$ is irreducible. For a sphere $\Sigma \subset S^3 - K$, by Alexander's thm, Σ bounds balls on either side in S^3. By connectivity, one doesn't intersect K, so Σ bounds a ball in $S^3 - K$.
2. $\pi_2(S^3 - K) = 0$. If it weren't, the Sphere Thm gives emb. $S^2 \cong \Sigma \subset S^3 - K$ (no $\mathbb{R}P^2$ since ori.). But irreducible $\Rightarrow$ $[\Sigma] = 0$!!
3. $S^3 - K$ is a $K(\pi, 1)$. $\pi_2(\widetilde{S^3 - K}) = 0$. $\widetilde{S^3 - K}$ is non-compact mfld, so $H_n(\widetilde{S^3 - K}) = 0$ for $n \geq 3$. Hurewicz $\Rightarrow$ $\pi_n(\widetilde{S^3 - K}) = 0$ for $n \geq 3$. Hence $\pi_n(S^3 - K)$ trivial if $n \neq 1$.

So knots complements with isomorphic π_1's are homotopy equivalent

 and  have homotopy equiv. but not homeomorphic complements.

* HNN extensions

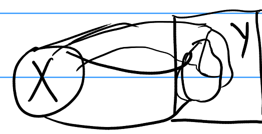
A a gp, B, B' subgps with $\varphi : B \to B'$ isomorphism

$$A *_{\varphi} := \langle A, t \mid \forall b \in B,\ tbt^{-1} = \varphi(b) \rangle$$

t "stable element"

Or, $B \underset{i_2}{\overset{i_1}{\rightrightarrows}} A$, $\quad A *_B = \langle A, t \mid \forall b \in B,\ t\, i_1(b) t^{-1} = i_2(b) \rangle$

Show up in mapping tori.



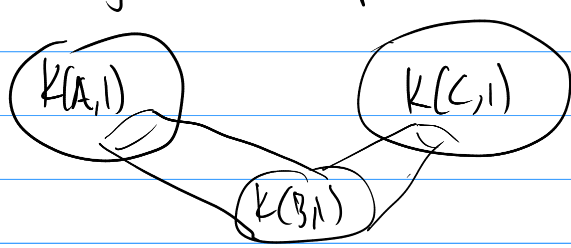
$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ gluing maps

$\pi_1(\text{m.torus}) = \langle \pi_1(Y), t \mid \forall b \in \pi_1(X),\ t f_{1*}(b) t^{-1} = f_{2*}(b) \rangle$.

This is a $K(A *_B, 1)$ if $X = K(B,1)$, $Y = K(A,1)$, f_{1*}, f_{2*} injections.

"nontrivial" if $B \neq A$ when $B \hookrightarrow A$ and $B \hookrightarrow A *_B$

* Amalgamated free products



is a $K(A *_B C, 1)$

"nontriv" means $B \neq A$, $B \neq C$ when $B \hookrightarrow A *_B C$

"Fundamental group of graph of groups"

* Groups acting on trees (Serre, "Trees")

Lemma A gp G admits a nontrivial decomposition $A *_B C$ or $A *_B$ iff G acts minimally on a nontrivial tree T without edge inversions, $B = \mathrm{Stab}_G\, e$

def minimally : for $v \in V(T)$, $\mathrm{hull}(Gv) = T$. Or, for $v, w \in V(G)$ exists g, g' s.t. path gv to $g'v'$ contains w.

edge inversion : $\exists\, e \in E(T)$ with $ge = e$ but swapped endpoints.
Fix by barycentric subdivision.

Pf Sps G acts on a tree T minimally w/o edge inversions.
Let $e \in E(T)$, replace T with collapse of each component of $T - G\,\mathrm{int}(e)$, a new tree, T/G is either or

Let u, v be vtcs of e in T. $A = \mathrm{Stab}_G\, u$, $B = \mathrm{Stab}_G\, e$, $C = \mathrm{Stab}_G\, v$

- If T/G is , $G = A *_B C$ ($B = A \cap C$ & decompose by "rotations")
- If T/G is , $gu = v$ for some $g \in G$, $gAg^{-1} = C$ (decompose by "slides")

If were trivial : sps $A = B$. A acts transitively on edges incident to v.
$A = B \Rightarrow$ single edge. $T/G =$ impossible!
$T/G = \quad \Rightarrow T = \quad \Rightarrow u$ fixed by G, so not minimal!

Converse.

1. $G = A *_B C$. Build $K(G,1)$ as mapping cylinders from $B \hookrightarrow A$, $B \hookrightarrow C$
   Take universal cover, look at $\widetilde{K(B,1)}$'s as edges of tree (A, C joined along subcplx)
2. $G = A *_B$. Build from $B \rightrightarrows A$. Similar.

* Ends of groups

(Freudenthal 1931)

Let X be a locally finite CW complex.

There is an inverse system of $\pi_0(X-K_1) \leftarrow \pi_0(X-K_2)$ for all $K_1 \subset K_2$ compact.

<u>def</u> $\mathcal{E}(X) = \varprojlim_K \pi_0(X-K)$ is the set of <u>ends</u> of X.

$\mathcal{E}(X)$ has a topology from giving each $\pi_0(X-K)$ the discrete topology.

Recall: $\varprojlim_K$ is a subspace of $\prod_K$.

Basis: choose $K_1, \dots, K_n \subset X$ compact, take ends that agree on these.
Or, since $K = K_1 \cup \dots \cup K_n$ is compact, just: ends that agree on some compact K.

Each $\pi_0(X-K)$ is finite, so $\mathcal{E}(X)$ is compact.
Can form "Freudenthal compactification"

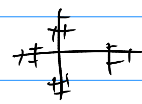
<u>ex</u>

$\mathbb{R}$: 2 ends

$\mathbb{R}^+$: 1 end

$\mathbb{R}^2$: 1 end

B^2 : 0 ends

 : ∞ ends (Cantor set)

For a finitely generated group G with a Cayley graph X, the <u>ends</u> $\mathcal{E}(G) := \mathcal{E}(X)$. Indep. of enerating set, up to homeo!

Homological version for X a locally finite conn. graph:

Define $C_e^*(X)$ by $0 \longrightarrow C_c^*(X; \mathbb{Z}/2\mathbb{Z}) \to C^*(X; \mathbb{Z}/2\mathbb{Z}) \to C_e^*(X) \to 0$

cochains with compact support

$C_e^0(X)$ is subsets of vertices V of X mod finite subsets, δV = edges between V & $1-V$

$H_e^0(X)$ is subsets V where δV is finite.

<u>Thm</u> $|\mathcal{E}(X)| = \dim H_e^0(X)$ (where all ∞'s are considered to be same)