Student 3-manifold seminar — Prime decomposition pt 2

Recall: a 3-manifold M is

- prime if separating S^2's bound a B^3
- irreducible if S^2's bound a B^3

The only closed orientable prime non-irreducible 3-mfld is $S^1 \times S^2$.

def A prime decomposition of connected orientable $M \not\cong S^3$ is $M = M_1 \# \cdots \# M_n$ with each M_i prime and $\not\cong S^3$.

Thm (Kneser 1929) For $M \not\cong S^3$ compact conn. ori., M has a prime decomp.

Pf Fix a finite triangulation τ of M. Let $t = \#$ 3-simplices, and suppose $M = M_1 \# \cdots \# M_n$ with each $M_i \not\cong S^3$. We will show that $n \leqslant 6t + \operatorname{rank} H_1(M; \mathbb{Z}/2\mathbb{Z})$. Thus: when n is maximal, each M_i is prime.

Can assume M has no nonseparating spheres: if so, has an $S^1 \times S^2$ summand (take arc α connecting comps of $\partial \nu(S)$, $\partial \nu(\alpha \cup S) \not\cong S^2$ and $\nu(\alpha \cup S) \cong S^1 \times S^2 - B^3$). $M = M' \# S^1 \times S^2 \Rightarrow H_1(M; \mathbb{Z}/2\mathbb{Z}) = H_1(M'; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$, so at most $\operatorname{rank} H_1(M; \mathbb{Z}/2\mathbb{Z})$ $S^1 \times S^2$'s in decomp. Remove them all.

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Let $S \subset M$ be a system of spheres for the decomposition.
(The components of $M - S$ corresp. to M_i's ; none are punctured S^3's)
Put S into general position w.r.t. τ:

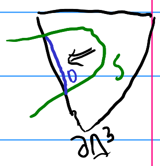
- S avoids τ^0
- S intersects edges τ^1 transversely at pts
- S intersects faces τ^2 transv. along arcs and circles

1) For Δ^3 a 3-simplex in τ, can make $S \cap \Delta^3$ a collection of disks

a) Any S^2 in $S \cap \Delta^3$ bounds a ball (Alexander's thm), !!

$\Rightarrow$ each component of $S \cap \Delta^3$ meets $\partial \Delta^3$

b) Consider $S \cap \partial \Delta^3$, a collection of circles.
Take innermost, bounding a disk $D \subset \Delta^3$ "near $\partial \Delta^3$"



- If ∂D bounds a disk D' in $S \cap \Delta^3$, $D \cup D'$ bounds a ball in Δ^3 disjoint from S. Can isotope D' of S to D.
Forget about this loop and go back to (b)

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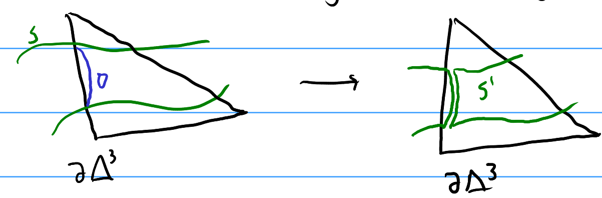
"embedded disk surgery"

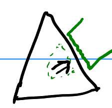
- If not, <u>compress</u> S along D to get S'



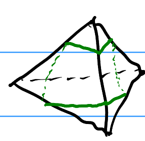
This is a new decomp. of M, with some $M_i = M_i' \# M_i''$.
since M has no non-sep. spheres

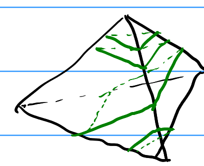
- If M_i' or M_i'' is S^3, can throw away a sphere in S' to yield same decomp.
- Else, WLOG use S' instead ($n+1 > n$ after all)

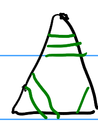
2) For each 2-simplex Δ^2, make $S \cap \Delta^2$ be only arcs between distinct faces

a) If , there is innermost such, bounding a disk in Δ^2.
Isotope along disk: 

b) If , both sides bound disks. Alexander's thm $\Rightarrow$ bounds B^3 !!

(At this point, S is nearly a <u>normal surface</u>. Would need to eliminate 's)

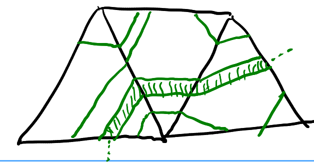


Faces look like  now.

For a Δ^3 in $\mathcal{T}$, consider $\partial\Delta^3 - S$, a collection of planar surfaces with $\sum \chi = 2$.

- Each disk contains a vtx of Δ^3, so ≤ 4 such
  Can enumerate all kinds of circles on $\partial\Delta^3$ (3, 4, or 8 edges)
- $\chi(\geq 3\times$ punctured $S^2) < 0$, so ≤ 2 such
- $\chi(\text{annulus}) = 0$, so any number

A <u>good</u> surface is an annulus w/o a vtx of Δ^3. At most 6 bad sfcs.
Each good annulus bounds two disks in $S \cap \Delta^3$, which together bound an I-bundle over a disk.



A component of $M-S$ made exclusively of these I-bundle pieces is an I-bundle. At most $6t$ aren't of this type.
The I-bundle has 1 or 2 boundary S^2's
- 1 S^2: it's $\mathbb{R}P^3 - B^3$ (twisted I-bdl over $\mathbb{R}P^2$). So some $M_i \cong \mathbb{R}P^3$, contributing a factor to $H_1(M;\mathbb{Z}/2\mathbb{Z})$.
- 2 S^2's: it's $S^2 \times I$. But then some $M_i \cong S^3$!!

Hence $n \leq 6t + \operatorname{rank} H_1(M;\mathbb{Z}/2\mathbb{Z})$. ∎

<u>Thm</u> (Milnor 1962) If $M \not\cong S^3$ is compact conn ori 3-mfld and $M = P_1 \# \cdots \# P_n \# a(S^1 \times S^2)$ and $M = Q_1 \# \cdots \# Q_m \# b(S^1 \times S^2)$ are prime decompositions with P_i's and Q_i's irreducible, then $m=n$, $a=b$, and the Q_i's are a permutation of the P_i's.

<u>Pf</u> Let S be a system of spheres representing the first prime decomp., along with nonsep. spheres reducing the $S^1 \times S^2$'s.
Let T be same but for second. Put S, T into general position ($S \pitchfork T$).
If $S \cap T \neq \emptyset$, let $D \subset T$ be a disk bounding innermost circle of $S \cap T$ on T.
Compress S along D, removing intersection, though now S has an additional S^3 factor. Repeat until $S \cap T = \emptyset$.
Now, add spheres of T to S, adding S^3 factors.
Eventually, $T \subset S$, so S represents <u>both</u> decompositions (along with S^3's)
Hence $m=n$ and Q_i's are perms of P_i's.
With $N = P_1 \# \cdots \# P_n$, $N \# a(S^1 \times S^2) = M = N \# b(S^1 \times S^2)$

$$\Rightarrow \quad H_1(N) \oplus \mathbb{Z}^a = H_1(M) = H_1(N) \oplus \mathbb{Z}^b$$

$$\Rightarrow \quad a = b.$$

∎

Caution: decomposition spheres are not isotopic! Consider (OO) vs OOO
(Decomposition is 2-surgery. If $M = \partial W$, a 4-mfld, corr. to attaching a 1-handle. Unique up to handle slides: O O $\longrightarrow$  ; $S^1 \times S^2$ though *is* a 1-handle, not a prime summand!)

$= P_1 \# P_2 \# P_3 \# 2(S^1 \times S^2)$

end of lecture

For nonorientable M: $S^1 \tilde{\times} S^2$ is nonorientable prime and not irreducible,

and $N \# (S^1 \tilde{\times} S^2) \cong N \# (S^1 \times S^2)$ iff N nonorientable

Prop If $p: \tilde{M} \to M$ is covering space with $\tilde{M}$ irred., so is M.

Pf Let $S \subset M$ be sphere. $p^{-1}(S)$ is disjoint spheres, each bounds a ball.
Let $\tilde{S} \subset p^{-1}(S)$ be innermost: bounds ball $B \subset \tilde{M}$ s.t. $B \cap p^{-1}(S) = \tilde{S}$.
$p|_B : B \to p(B)$ is a covering space $(\cdots)$
Since it is single-sheeted on $\tilde{S}$, it is a homeo., so S bounds ball $p(B)$. ∎

ex $L(p,q) = S^3/\mathbb{Z}_q$ where $S^3 \subset \mathbb{C}^2$ and $\mathbb{Z}_q$ gen by $(z_1, z_2) \mapsto (e^{2\pi i/q} z_1, e^{2\pi i p/q} z_2)$
Includes $\mathbb{R}P^3$

ex $M = S^1 \times (\text{cpct sfc})$ or M a (cpct sfc)-bdle over S^1. Then $\tilde{M} \cong \mathbb{R}^3$ if sfc $\neq S^2$ or $\mathbb{R}P^2$

nonex $\exists$ 2-sheet cover $S^1 \times S^2 \to \mathbb{R}P^3 \# \mathbb{R}P^3$ $\quad ((x,y) \sim (\text{refl}(x), -y))$

Aside One may split M with ∂ along properly embedded disks $(D^2, S^1) \hookrightarrow (M, \partial M)$
Boundary connect sum decomps

Aside One may split non-ori. M's along 2-sided $\mathbb{R}P^2$'s $\quad$ ($\partial \nu \mathbb{R}P^2$ has 2-comps)

* Heegaard splittings

A *genus-g handlebody* H is a cpct ori 3-mfld s.t. $\partial H = \Sigma_g$ and $\exists$ collection D of properly embedded disks s.t. $H - D \cong B^3$
actually: compressed along D.

(I.e., $H \cong \natural_g B^2 \times S^1$, g-fold boundary connect sum)

A *Heegaard splitting* of a closed, orientable 3-mfld M is a closed ori sfc $S \subset M$ s.t. $M - S$ is two handlebodies.

(Heeg. 1898)

<u>Prop</u> Every closed, ori. 3-mfld has a Heegaard splitting.

<u>Pf</u> Let τ be a triangulation. $\partial\nu(\tau^1)$ is a closed orientable surface.
$\nu(\tau^1)$ is a handlebody, and so is $M - \nu(\tau^1)$. ▨

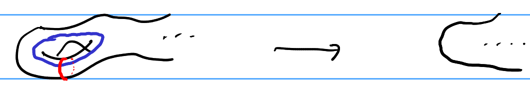
<u>def</u> The <u>Heegaard genus</u> of M is the minimal genus of any Heeg-spl.

<u>Prop</u> S^3 is the only mfld w/ Heeg. genus $=0$.

<u>Pf</u> Genus $0 \Rightarrow M = S^3 \# S^3 \cong S^3$. ▨

<u>ex</u> $L(p,q)$ has genus $=1$.

<u>destabilization</u>:



↑ bounding disks on either side

<u>Thm</u> (Reidemeister-Singer) Any two Heeg. spl. of M are related by stabilizations.

A Heegaard <u>reducing sphere</u> Σ intersects S in an essential separating loop.

<u>Thm</u> If $M = M_1 \# M_2$ is nontrivial, then there is a Heeg. reducing sphere.

$\Rightarrow g(M) = g(M_1) + g(M_2)$

<u>Prop</u> (Waldhausen) Every $g>0$ H.S. of S^3 can be destabilized.
That is, every H.S. is <u>standard</u>.