Student 3-manifold Seminar

math.berkeley.edu/~kmill/st3ms

A theme: study 3-manifolds through their incompressible surfaces
(or: combinatorial "cut-and paste" methods)

This seminar: get acquainted with the 1930s–1980s

Some possible topics

- Decompositions: Prime, JSJ, Heegaard splittings
- Hierarchies: Haken, sutured manifolds
- Geometric classifications and Thurston's conjecture (theorem)
- Submanifolds from group theory: loop thm, sphere thm
- Normal surface theory & recognition algorithms
- Dehn surgery, surgery diagrams, branched covers
- (PL vs smooth vs triangulations ?)
- Thin position arguments (Abby Thompson)

References: Hempel, Jaco, Thurston, Hatcher, Calegari, Rolfsen,
Rourke & Sanderson (PL)

Prime decompositions

* What is a 3-mfld?

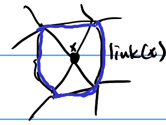
Recall: An <u>n-manifold</u> M is a 2^{nd}-countable Hausdorff space that is locally homeo. to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$

$\partial M =$ all pts on $\partial \mathbb{R}^3_+$, is empty or is an $(n-1)$-mfld

M is <u>closed</u> if $\partial M = \emptyset$ and M is compact.

- Piecewise-linear (PL) manifolds

An n-dim simplicial complex is <u>combinatorial</u> if for each vtx x, link(x) is S^{n-1} or B^{n-1}



A <u>triangulation</u> of M is a homeomorphism $\tau \to M$ with τ a simplicial complex. Two triangulations are <u>compatible</u> if they have a common subdivision. A <u>PL structure</u> is a maximal set of compatible combinatorial triangulations. A <u>PL manifold</u> is a manifold with a PL structure.

A map $f: M_1 \to M_2$ is PL if there are triangulations $\tau_i \to M_i$ such that

$$\begin{tikzcd} \tau_1 \arrow[r, dashed, "f'"] \arrow[d] & \tau_2 \arrow[d] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

f' is simplicial.

<u>Thm</u> (Bing & Moise, 1950s) Every 3-mfld has a unique PL structure.

Every smooth 3-mfld has a unique PL structure from a PD triangulation.

PL = smooth in dimension 3

Submanifolds are different in Top vs. PD. <u>Ex</u> Alexander horned sphere in S^3

Need to be from a triangulation.

We can reason about PL mflds as if they were smooth:

- $\exists$ tubular nbhds (embs. of normal bundles) $\nu(S)$ ("regular nbhds" in PL)
- isotopies of submflds $\Rightarrow$ ambient isotopy
- transversality by perturbations

* Decomposition

analytic ("break apart")

<u>def</u> If $S \subset M$ (conn.) is a separating sphere, write $M = M_1 \# M_2$ ("connect sum") with M_1 and M_2 from filling the boundaries of $M - \nu(S)$ with B^3's
(Fact: every $S^2 \xrightarrow{\cong} S^2$ is diffeo to id or refl., so only one way to glue B^3's)

M is <u>prime</u> if $M = M_1 \# M_2 \Rightarrow M_1 = B^3$ or $M_2 = B^3$
("separating S^2's bound balls")

synthetic ("put together")

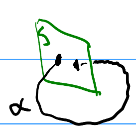
Fact: every pair of B^3's in a conn. 3-mfld are isotopic (up to ori.-reversal), so $M_1 \# M_2$ is well-defined op. with choice of B^3 orientations.

<u>Thm</u> (Alexander, 1924) Every embedded S^2 in $\mathbb{R}^3$ bounds an emb. B^3.
Sketch: slice the S^2 up using a Morse fn ...

<u>Cor</u> S^3 is prime.
<u>Pf</u> Let $S \subset S^3$ be an S^2, $x \in S^3 - S$. $S^3 - x \cong \mathbb{R}^3$, so Alex.'s thm $\Rightarrow$ S bounds ball.

<u>Def</u> M is <u>irreducible</u> if every emb. S^2 bounds a ball.
(irred. $\Rightarrow$ prime)

<u>Prop</u> If M is conn., oriented, prime 3-mfld that is not irred., $M \cong S^1 \times S^2$.
<u>Pf</u> Let $S \subset M$ be a nonsep. sphere (if were sep., prime $\Rightarrow$ bounds ball).
Let α be an arc from one side of S to other. $U = \nu(\alpha) \cup \nu(S)$ is $S^1 \times S^2 - B^3$, and ∂U is a separating sphere. Hence $M \cong M' \# S^1 \times S^2$.



<u>Cor</u> The only non-irred. conn., ori, prime 3-mfld is $S^1 \times S^2$.
(from prop. on next page)

Prop $S^1 \times S^2$ is prime.

Pf Suppose S is a separating sphere, U, V comps of $S^1 \times S^2 - \nu(S)$.
van Kampen: $\mathbb{Z} = \pi_1(S^1 \times S^2) = \pi_1(U) * \pi_1(V)$, so WLOG $\pi_1(V) = 1$.
Universal cover $\widetilde{S^1 \times S^2} \cong \mathbb{R}^3 - \{0\}$, V lifts to $\tilde{V}$, a diffeo. copy.
Alex thm: $\partial \tilde{V}$ bounds ball in $\mathbb{R}^3$. $\partial \tilde{V}$ bounds $\tilde{V}$ in $\mathbb{R}^3 - \{0\} \Rightarrow \tilde{V} \cong B^3$.
Hence $V \cong B^3$.

Def A <u>prime decomposition</u> of conn. ori. $M \neq S^3$ is $M = M_1 \# \cdots \# M_n$ with each M_i prime and not S^3.

Thm (Kneser 1929) Let $M \neq S^3$ be compact, conn., ori.. Then M has a prime decomp.

Pf Fix a finite triangulation $T \to M$, let $t = \#$ 3-simplices, and let $M = M_1 \# \cdots \# M_n$ with each $M_i \neq S^3$.
We will show that $n \leq k(M) = 6t + \operatorname{rk} H_1(M; \mathbb{Z}/2\mathbb{Z})$.
Thus, connect sum decomposition eventually yields prime factors. ... <u>Next time</u> ...

What is the difficulty? If M is compact, $\pi_1(M)$ is finitely generated.
A splitting gives $\pi_1(M) = \pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$ by van Kampen thm.
If $\pi_1(M_1)$ and $\pi_1(M_2)$ nontrivial, then both groups have fewer generators.
So: there is a finite splitting $M = M_1 \# \cdots \# M_n$ with each M_i prime or simply connected. Suppose M is closed and oriented.

$\pi_1(M_i) = 0 \Rightarrow \cdots \Rightarrow$ there is a homotopy equivalence $S^3 \to M_i$

Poincaré conj. $\Rightarrow M_i \cong S^3$, but that seems heavy handed!
$6t$ is an upper bound on # generators, and even so $\dim H_1(M_i; \mathbb{Z}/2\mathbb{Z})$ $\mathbb{RP}^3$'s will be allowed for in the proof.

Also next time:

Thm (Milnor 1962) If $M \neq S^3$ is cpct., conn., ori 3-mfld and $M = P_1 \# \cdots \# P_n \# a(S^1 \times S^2)$ and $M = Q_1 \# \cdots \# Q_m \# b(S^1 \times S^2)$ with P_i's and Q_i's prime and irred., then $a = b$, $n = m$, and the Q_i's are a permutation of the P_i's.