Three-Dimensional Manifolds

Michaelmas Term 1999

Prerequisites

Basic general topology (eg. compactness, quotient topology)
Basic algebraic topology (homotopy, fundamental group, homology)

Relevant books

Armstrong, *Basic Topology* (background material on algebraic topology)
Hempel, *Three-manifolds* (main book on the course)
Stillwell, *Classical topology and combinatorial group theory* (background material, and some 3-manifold theory)

§1. Introduction

Definition. A *(topological) n-manifold* M is a Hausdorff topological space with a countable basis of open sets, such that each point of M lies in an open set homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. The *boundary* ∂M of M is the set of points not having neighbourhoods homeomorphic to $\mathbb{R}^n$. The set $M - \partial M$ is the *interior* of M, denoted $\mathrm{int}(M)$. If M is compact and $\partial M = \emptyset$, then M is *closed*.

In this course, we will be focusing on 3-manifolds. Why this dimension? Because 1-manifolds and 2-manifolds are largely understood, and a full 'classification' of n-manifolds is generally believed to be impossible for $n \geq 4$. The theory of 3-manifolds is heavily dependent on understanding 2-manifolds (surfaces). We first give an infinite list of closed surfaces.

Construction. Start with a 2-sphere S^2. Remove the interiors of g disjoint closed discs. The result is a compact 2-manifold with non-empty boundary. Attach to each boundary component a 'handle' (which is defined to be a copy of the 2-torus T^2 with the interior of a closed disc removed) via a homeomorphism between the boundary circles. The result is a closed 2-manifold F_g of *genus* g. The surface F_0 is defined to be the 2-sphere S^2.

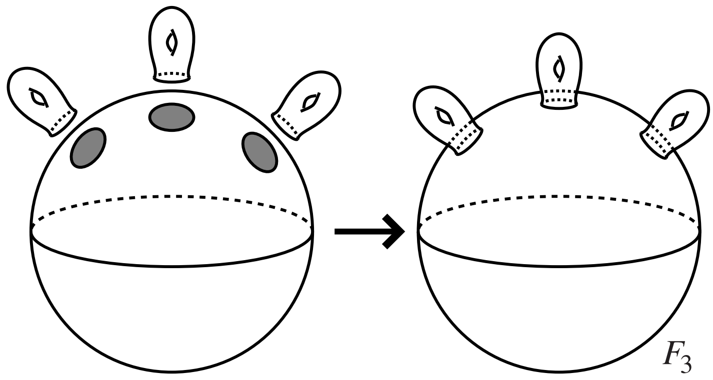


Figure 1.

Construction. Start with a 2-sphere S^2. Remove the interiors of h disjoint closed discs ($h \geq 1$). Attach to each boundary component a Möbius band via homeomorphisms of the boundary circles. The result is a closed 2-manifold N_h.

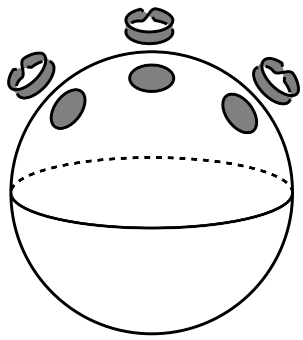


Figure 2.

Exercise. N_1 is homeomorphic to the real projective plane P^2.

Theorem 1.1. (Classification of closed 2-manifolds) *Each closed 2-manifold is homeomorphic to precisely one F_g for some $g \geq 0$, or one N_h for some $h \geq 1$.*

This is an impressive result. There is a similar result for compact 2-manifolds with boundary.

Theorem 1.2. (Classification of compact 2-manifolds) *Each compact 2-manifold is homeomorphic to precisely one of $F_{g,b}$ or $N_{h,b}$, where $g \geq 0$, $b \geq 0$ and $h \geq 1$, and $F_{g,b}$ (resp. $N_{h,b}$) is homeomorphic to F_g (resp. N_h) with the interiors of b disjoint closed discs removed.*

The surface $F_{0,1}$ is a disc D^2, $F_{0,2}$ is an *annulus* and $F_{0,3}$ is a *pair of pants*;

the surfaces $F_{0,i}$ $(i \geq 1)$ are the *compact planar surfaces*.

There is in fact a classification of non-compact 2-manifolds, but the situation is significantly more complicated than in the compact case. In dimensions more than two, it is usual to concentrate on compact manifolds (which are usually hard enough). Below are some examples of non-compact 2-manifolds (without boundary) that exhibit a wide range of behaviour.

Examples. (i) $\mathbb{R}^2$.

(ii) The complement of a finite set of points in a closed 2-manifold.

(iii) $\mathbb{R}^2 - (\mathbb{Z} \times \{0\})$.

(iv) Glue a countable collection of copies of $F_{1,2}$ 'end-to-end'.

(v) Start with an annulus. Glue to each boundary component a pair of pants. The resulting 2-manifold has four boundary components. Glue to each of these another pair of pants. Repeat indefinitely.

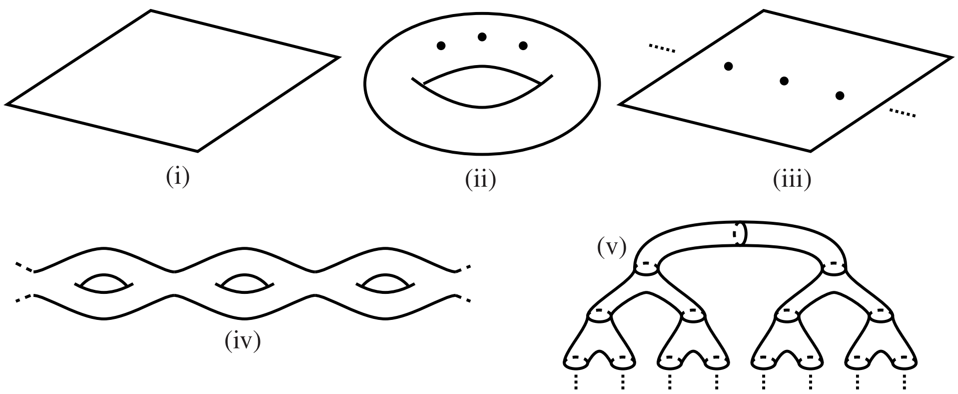


Figure 3.

It is quite possible that there is some sort of classification of compact 3-manifolds similar to the 2-dimensional case, but inevitably much more complicated. The simplest closed 3-manifold is the 3-sphere, which is most easily visualised as $\mathbb{R}^3$ 'with a point at infinity'.

Exercise. Prove that, for any point $x \in S^3$, $S^3 - \{x\}$ is homeomorphic to $\mathbb{R}^3$.

Construction. Let X be a subset of S^3 homeomorphic to the solid torus $S^1 \times D^2$. Then $S^3 - \text{int}(X)$ is a compact 3-manifold, with boundary a torus. Note that there are many possible such X in S^3 (one is given in Figure 4), and hence there are many such 3-manifolds.

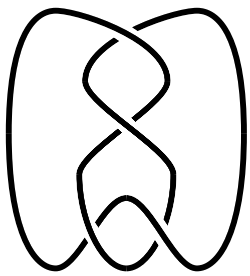


Figure 4.

Despite the large number of different 3-manifolds, they have a well-developed theory.

Definition. Let M_1 and M_2 be two oriented 3-manifolds. (The definition of an oriented manifold will be given in the next section.) Pick subsets B_1 and B_2 homeomorphic to closed 3-balls in the interiors of M_1 and M_2. Let $M_1 \# M_2$ be the manifold obtained from $M_1 - \mathrm{int}(B_1)$ and $M_2 - \mathrm{int}(B_2)$ by gluing ∂B_1 and ∂B_2 via an orientation-reversing homeomorphism. Then $M_1 \# M_2$ is the *connected sum* of M_1 and M_2.

The resulting 3-manifold $M_1 \# M_2$ is in fact independent of the choice of B_1, B_2 and orientation-reversing homeomorphism $\partial B_1 \to \partial B_2$. The 3-sphere is the union of two 3-balls glued along their boundaries. When one is forming $M \# S^3$ for any 3-manifold M, we may assume that one of these 3-balls is used in the definition of connected sum. Hence, $M \# S^3$ is obtained from M by removing a 3-ball and then gluing another back in. Hence, $M \# S^3$ is homeomorphic to M. A 3-manifold M is *composite* if it is homeomorphic to $M_1 \# M_2$, for neither M_1 nor M_2 homeomorphic to S^3; otherwise it is *prime*.

Here is an example of a theorem in this course.

Theorem 1.3. (Topological rigidity) *Let M_1 and M_2 be closed orientable prime 3-manifolds which are homotopy equivalent. Suppose that $H_1(M_1)$ and $H_1(M_2)$ are infinite. Then M_1 and M_2 are homeomorphic.*

The theorem can be false:

- if M_1 and M_2 are not prime,
- if $H_1(M_1)$ and $H_1(M_2)$ are finite,
- if M_1 and M_2 have non-empty boundary, or
- if M_1 and M_2 are non-compact.

Example. The following is a construction of two compact orientable prime 3-manifolds M_1 and M_2, with non-empty boundary, that are homotopy equivalent but not homeomorphic. Pick two disjoint simple closed curves in a torus T^2, bounding disjoint discs in T^2. Attach to each curve a copy of $F_{1,1}$ along the boundary curve of $F_{1,1}$. The resulting space X will be homotopy equivalent to both M_1 and M_2.

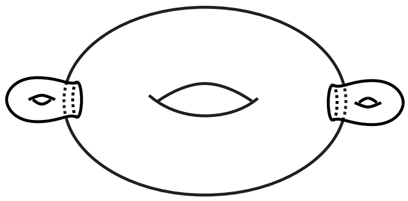


Figure 5.

We construct M_1 and M_2 by 'thickening' T^2 and the two copies of $F_{1,1}$ to $T^2 \times [0, 1]$ and two copies of $F_{1,1} \times [0, 1]$. We build M_1 by gluing the two copies of $\partial F_{1,1} \times [0, 1]$ to disjoint annuli in $T^2 \times \{0\}$ (the annuli separating off disjoint discs in $T^2 \times \{0\}$). Note that M_1 is a 3-manifold with ∂M_1 being three tori and a copy of F_3. We construct M_2 similarly, except we attach one of the two copies of $\partial F_{1,1} \times [0, 1]$ to $T^2 \times \{0\}$ and one to $T^2 \times \{1\}$. The resulting manifold M_2 has ∂M_2 being two tori and two copies of F_2. Hence, M_1 and M_2 are not homeomorphic, but they are both homotopy equivalent to X. (We cannot at this stage prove that they are prime, but this is in fact true.)

However, it is widely believed that (in a sense that can be made precise) 'almost all' homotopy equivalent closed 3-manifolds are in fact homeomorphic. A special case of this is the following, which is one of the most famous unsolved conjectures in topology.

Poincaré Conjecture. *A 3-manifold homotopy equivalent to S^3 is homeomorphic to S^3.*

§2. Which category?

In manifold theory, it is very important to specify precisely which 'category' one is working in. For example, one can deal not only with topological manifolds, but also smooth manifolds (which we will not define) and piecewise-linear (pl) manifolds, which are defined below. It turns out that 3-manifold theory often takes place in the pl setting.

Definition. The *n-simplex* is the set

$$\Delta^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \dots + x_{n+1} = 1 \text{ and } x_i \geq 0 \text{ for all } i\}.$$

The *dimension* of Δ^n is n. A *face* of an n-simplex Δ^n is a subset of Δ^n in which some co-ordinates are set to zero. A face of dimension zero is a *vertex*.

Definition. A *simplicial complex* is the space K obtained from a collection of simplices by gluing their faces together via linear homeomorphisms, such that any point of K has a neighbourhood intersecting only finitely many simplices.

Remark. This definition is more general than the usual definition of a simplicial complex, where one insists that each collection of points forms the vertices of at most one simplex.

Note. The underlying space of a simplicial complex is compact if and only if it has finitely many simplices.

Definition. A *triangulation* of a space M is a homeomorphism from M to some simplicial complex.

Example. The space obtained from two copies of Δ^n by identifying their boundaries using the identity map is a simplicial complex. It forms a triangulation of the n-sphere.

Definition. A *subdivision* of a simplicial complex K is another simplicial complex L with the same (i.e. homeomorphic) underlying space as K, where each simplex of L lies in some simplex of K in such a way that the inclusion map is affine.

Definition. A map $f: K \to L$ between simplicial complexes is *pl* if there exists subdivisions K' and L' of K and L so that f sends vertices of K' to vertices of L', and sends each simplex of K' linearly (but not necessarily homeomorphically) onto a simplex of L'.

Thus, by definition, there exists a pl homeomorphism between two simplicial complexes if and only if they have a common subdivision.

Exercise. The composition of two pl maps is again pl. Hence, simplicial complexes and pl maps form a category.

Definition. A *pl n-manifold* is a simplicial complex in which each point has a neighbourhood pl homeomorphic to the n-ball

$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \leq 1 \text{ for each } i\}$$

(with a standard triangulation).

An important fact that simplifies much of 3-manifold theory is the following theorem, due to Moise.

Theorem 2.1. *A topological 3-manifold possesses precisely one smooth structure (up to diffeomorphism) and precisely one pl structure (up to pl homeomorphism).*

This theorem is false in dimensions greater than three. When studying 3-manifold theory, however, it does not matter which category one pursues it from. For simplicity, we will now work entirely in the pl category without explicitly stating this. **Thus, all manifolds will be pl, and all maps will be pl.**

We now introduce a couple of concepts that are probably familiar, in a pl setting.

Orientability

Definition. An *orientation* on an n-simplex is an equivalence class of orderings on its vertices, where we treat distinct orderings as specifying the same orientation if and only if the orderings differ by an even permutation. If the vertices are ordered as $v_0, \dots, v_n$ (say), then we write $[v_0, \dots, v_n]$ for this orientation. We write $-[v_0, \dots, v_n]$ for the other orientation. The orientation $[v_0, \dots, v_n]$ *induces* the orientation $(-1)^i[v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$ on the face opposite v_i.

Definition. An *orientation* on an n-manifold M is a choice of orientation on each n-simplex of M, such that, if σ is any $(n-1)$-simplex adjacent to two n-simplices, then the orientations that σ inherits from these simplices disagree. The manifold is then *oriented.* If a triangulation of a manifold does not admit an orientation, then the manifold is *non-orientable*.

Note. A compact n-manifold M is orientable if and only if $H_n(M, \partial M) = \mathbb{Z}$. In this case, an orientation is a choice of generator for $H_n(M, \partial M)$. Hence, orientability is independent of the choice of triangulation for compact manifolds (and in fact for all manifolds).

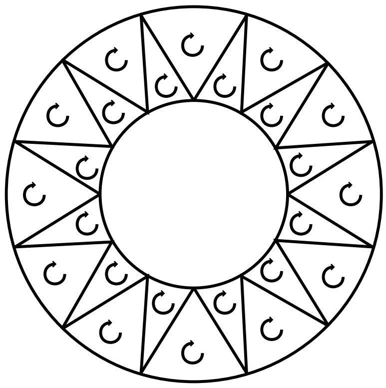


Figure 6.

Examples. The Möbius band M is non-orientable, whereas the annulus A is orientable. See Figure 6, where the arrows on each 2-simplex specify an orientation on that 2-simplex in the obvious way. Note that M and A are homotopy equivalent.

SUBMANIFOLDS

Note that D^k sits inside D^n for $k \leq n$, by setting the co-ordinates $x_{k+1}, \ldots, x_n$ to zero.

Definition. A *submanifold* X of a pl manifold M is a subset which is simplicial in some subdivision of M, such that each point of X has a neighbourhood N and a pl homeomorphism $(N, N \cap X) \to (D^n, D^k)$. Note that this implies that $\partial X = X \cap \partial M$.

Definition. A map $X \to M$ between simplicial complexes is an *embedding* if it is a pl homeomorphism onto its image. It is a *proper embedding* if M is a manifold and the image of X is a submanifold of M.

Example. A 1-dimensional submanifold of S^3 is a *link*. If it is connected, it is a *knot*. If K is a knot in S^3 that does not bound a disc and we 'cone' the pair

(S^3, K), the result is a 2-disc embedded in the 4-ball, but not properly embedded.

Exercise. Show that if S is a surface embedded in a 3-manifold M such that $S \cap \partial M = \partial S$, then S is properly embedded. (You will need to know that any circle embedded in S^2 is 'standard'.)

We will see that studying submanifolds of M will shed considerable light on the properties of M.

We will prove the following result in §6.

Proposition 2.2. *Let X be an orientable codimension one submanifold of an orientable manifold. Then X has a neighbourhood homeomorphic to $X \times [-1, 1]$, where $X \times \{0\}$ is identified with X, and where $(X \times [-1, 1]) \cap \partial M) = \partial X \times [-1, 1]$.*

Isotopies

Let M be a simplicial complex.

Definition. Two homeomorphisms $h_0 \colon M \to M$ and $h_1 \colon M \to M$ are *isotopic* if there is a homeomorphism $H \colon M \times [0, 1] \to M \times [0, 1]$ such that, for all i, $H|_{M\times\{i\}}$ is a homeomorphism onto $M \times \{i\}$, and so that $H|_{M\times\{0\}} = h_0$ and $H|_{M\times\{1\}} = h_1$.

Remark. It is possible to impose a topology on the set $\mathrm{Homeo}(M, M)$ of all (pl) homeomorphisms $M \to M$, such that the path-components of $\mathrm{Homeo}(M, M)$ are precisely the isotopy classes.

Definition. Let K_0 and K_1 be subsets of M. They are *ambient isotopic* if there is a homeomorphism $h \colon M \to M$ that is isotopic to the identity and that takes K_0 to K_1.

Subsets of M that are ambient isotopic are, for almost all topological purposes, 'the same' and we will feel free to perform ambient isotopies as necessary.

§3. Incompressible surfaces

The majority of 3-manifold theory studies submanifolds of a 3-manifold M, and uses them to gain information about M. This is particularly fruitful because surfaces (i.e. 2-manifolds) are well understood. However, only certain surfaces embedded within M have any relevance. The most important of these are 'incompressible' and are defined as follows.

Definition. Let S be a properly embedded surface in a 3-manifold M. Then a *compression disc* D for S is a disc D embedded in M such that $D \cap S = \partial D$, but with ∂D not bounding a disc in S. If no such compression disc exists, then S is *incompressible*.

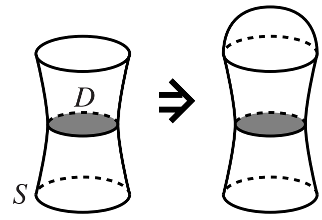


Figure 7.

Of course, a 2-sphere or disc properly embedded in a 3-manifold is always incompressible.

Remark. Suppose that D is a compression disc for S. We may assume that D lies in $\mathrm{int}(M)$. There is then a way of 'simplifying' S as follows. Essentially using Proposition 6.6 (see §2), we may find an embedding of $D \times [-1,1]$ in $\mathrm{int}(M)$ with $(D \times [-1,1]) \cap S = \partial D \times [-1,1]$. Then

$$S \cup (D \times \{-1,1\}) - (\partial D \times (-1,1))$$

is a new surface properly embedded in M. It is obtained *by compressing S along D*.

Denote the Euler characteristic of compact surface S by $\chi(S)$. Define the *complexity* of S to be the sum of $-\chi(S)$, the number of components of S and the number of 2-sphere components of S. Note that this number is non-negative. A compression to S reduces $-\chi(S)$ by two. It either leaves the number of components unchanged or increases it by one. It does not create any 2-sphere components, unless S is a torus or Klein bottle compressing to a 2-sphere. Hence, we have the following.

Lemma 3.1. *Compressing a surface decreases its complexity.*

We will occasionally abuse notation by 'compressing' along a disc D with $D \cap S = \partial D$, but with ∂D bounding a disc in S. Note that in this case, the complexity of the surface is left unchanged.

Definition. A compact orientable 3-manifold is *Haken* if it is prime and contains a connected orientable incompressible properly embedded surface other than S^2.

Note that every compact orientable prime 3-manifold M with non-empty boundary is Haken. For we may pick a disc in ∂M and push its interior into the interior of M so that the disc is properly embedded. This is a connected orientable incompressible properly embedded surface, as required. Of course, it is not a particularly interesting surface, but we will see later that, unless M is a 3-ball, other interesting surfaces also live in M.

Haken was a prominent 3-manifold topologist, and he was the first person to realize the importance of incompressible surfaces. (He also has a number of other mathematical accolades; for example, he proved the famous 4-colour theorem in graph theory.) Haken 3-manifolds are extremely well understood. For example, we will prove the following topological rigidity theorem.

Theorem 3.2. *Let M and M' be closed orientable 3-manifolds, with M Haken and M' prime. If M and M' are homotopy equivalent, then they are homeomorphic.*

Another major result which demonstrates the usefulness of incompressible surfaces is the following.

Theorem 3.3. *Let S be an orientable surface properly embedded in a compact prime orientable 3-manifold M. Then S is incompressible if and only if the map $\pi_1(S) \rightarrow \pi_1(M)$ induced by inclusion is an injection.*

In one direction (that π_1-injectivity implies incompressibility) this is quite straightforward, but the converse is difficult and quite surprising. We will prove this theorem later in the course.

We now demonstrate that Haken 3-manifolds are fairly common, by giving plenty of examples of incompressible surfaces in various manifolds.

Definition. A connected surface S properly embedded in a connected 3-manifold M is *non-separating* if $M - S$ is connected.

Lemma 3.4. *Let S be a surface properly embedded in a 3-manifold M. The following are equivalent:*

(i) S is non-separating;

(ii) there is a loop properly embedded in M which intersects S transversely in a single point;

(iii) there is a loop properly embedded in M which intersects S transversely in an odd number of points.

Proof. (i) $\Rightarrow$ (ii). Suppose that S is non-separating. Pick a small embedded arc intersecting S transversely. The endpoints of this arc lie in the same path-component of $M - S$, and so may be joined by an arc in $M - S$. The two arcs join to form a loop, which we may assume is properly embedded. This intersects S transversely in a single point.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). If S separates M into two components, any loop in M intersecting S transversely alternates between these components. Hence, it intersects S an even number of times. □

Example. The 3-torus $S^1 \times S^1 \times S^1$ contains a non-separating torus.

Proposition 3.5. *Let M be a prime orientable 3-manifold containing a non-separating 2-sphere S^2. Then M is homeomorphic to $S^2 \times S^1$.*

Proof. By Proposition 6.6, S^2 has a neighbourhood homeomorphic to $S^2 \times [-1, 1]$. Since S^2 is non-separating, there is a loop ℓ properly embedded in M intersecting S^2 transversely in a single point. For small enough $\epsilon > 0$, $\ell \cap (S^2 \times [-\epsilon, \epsilon])$ is a single arc. Using technology that we will develop in §6, $\ell - (S^2 \times [-\epsilon, \epsilon])$ has a neighbourhood in $M - (S^2 \times (-\epsilon, \epsilon))$ homeomorphic to a ball B such that

$B \cap (S^2 \times \{-\epsilon\})$ and $B \cap (S^2 \times \{\epsilon\})$ are two discs. Then, using an obvious product structure on B, $X = B \cup (S^2 \times [-\epsilon, \epsilon])$ is homeomorphic to $S^2 \times S^1$ with the interior of a closed 3-ball removed. Note that ∂X is a separating 2-sphere in M. Hence, since M is prime, this bounds a 3-ball B' in M. Then $M = X \cup B'$ is homeomorphic to $S^2 \times S^1$. □

A 3-manifold M is known as *irreducible* if any embedded 2-sphere in M bounds a 3-ball. Otherwise, it is *reducible*. By Proposition 3.5, an orientable reducible 3-manifold is either composite or homeomorphic to $S^2 \times S^1$.

Proposition 3.6. *Let M be an orientable prime 3-manifold containing a properly embedded orientable non-separating surface S. Then M is either Haken or a copy of $S^2 \times S^1$.*

Proof. If M contains a non-separating 2-sphere, we are done. If S is incompressible, we are done. Hence, suppose that S compresses to a surface S'. Then S' is orientable. By Lemma 3.3, there is a loop ℓ intersecting S transversely in a single point. By shrinking the product structure on $D \times [-1, 1]$ as in the proof of Proposition 3.5, we may assume that ℓ intersects $D \times [-1, 1]$ in arcs of the form $\{*\} \times [-1, 1]$. Hence, it intersects S' transversely in an odd number of points. So, at least one component of S' is non-separating. By Lemma 3.1, the complexity of this component is less than that of S. Hence, we eventually terminate with an incompressible orientable non-separating surface. □

Example. The above argument gives that any non-separating torus in $S^1 \times S^1 \times S^1$ is incompressible. (We need to know, in addition, that $S^1 \times S^1 \times S^1$ is prime.)

We will prove the following result in §7. In combination with Proposition 3.6, this provides examples of many Haken 3-manifolds.

Theorem 3.7. *Let M be a compact orientable 3-manifold. If $H_1(M)$ is infinite, then M contains an orientable non-separating properly embedded surface.*

The converse of Theorem 3.7 is also true. So this does not in fact create any more examples of Haken manifolds than Proposition 3.6. However, it is often more convenient to calculate the homology of a 3-manifold than to construct an explicit non-separating surface in it.

There is one notable 3-manifold that is not Haken.

Theorem 3.8. *The only connected incompressible surface properly embedded in S^3 is a 2-sphere. Hence, S^3 is not Haken.*

At the same time, we will prove.

Theorem 3.9. (Alexander's theorem) *Any pl properly embedded 2-sphere in S^3 is ambient isotopic to the standard 2-sphere in S^3. In particular, it separates S^3 into two components, the closure of each component being a pl 3-ball. Hence, S^3 is prime.*

Remark. The theorem is not true for topological embeddings of S^2 in S^3. Also, it is remarkable that the corresponding statement for pl or smooth 3-spheres in S^4 remains unproven.

Proof of Theorems 3.8 and 3.9. Let S be a connected incompressible properly embedded surface in S^3. We will show that S is ambient isotopic to the standard 2-sphere in S^3. Let p be some point in $S^3 - S$. Then $S^3 - p$ is pl homeomorphic to $\mathbb{R}^3$. Hence, S is simplicial in some subdivision of a standard triangulation of $\mathbb{R}^3$.

Claim. There is a product structure $\mathbb{R}^2 \times \mathbb{R}$ on $\mathbb{R}^3$, and an ambient isotopy of S, so that after this isotopy, the following is true: for all but finitely many $x \in \mathbb{R}$, $(\mathbb{R}^2 \times \{x\}) \cap S$ is a collection of simple closed curves, and at each of the remaining $x \in \mathbb{R}$, we have one 'singularity' of of one of the following forms:

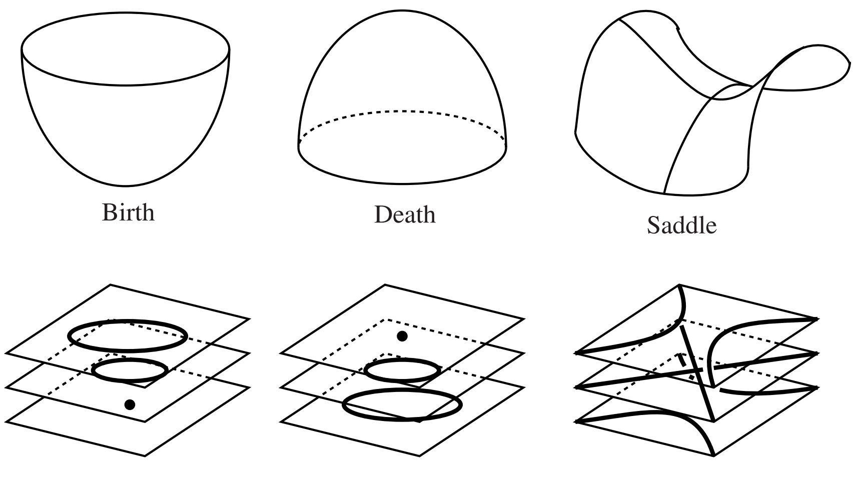


Figure 8.

Proof of claim. Each simplex in the triangulation of $\mathbb{R}^3$ is convex in $\mathbb{R}^3$. The set of unit vectors parallel to 1-simplices of S is finite. We take a product structure $\mathbb{R}^2 \times \mathbb{R}$, so that neither $\mathbb{R}^2 \times \{0\}$ nor $\{0\} \times \mathbb{R}$ contains any of these vectors. We may also assume that, for each x, $\mathbb{R}^2 \times \{x\}$ contains at most one vertex of S. When $\mathbb{R}^2 \times \{x\}$ does not contain a vertex of S, $(\mathbb{R}^2 \times \{x\}) \cap S$ is a collection of simple closed curves. Near the vertices of S, the singularities are a little more complicated than required, and hence we perform an ambient isotopy of S to improve the situation. Let ϵ be the length of the shortest 1-simplex in $\mathbb{R}^3$ that intersects S. Focus on a single vertex v of S. Let B be the polyhedron in $\mathbb{R}^3$ with vertices at precisely the points on the 1-simplices of $\mathbb{R}^3$ at distance $\epsilon/2$ from v. Then we may subdivide $\mathbb{R}^3$ further so that B is simplicial. Then $S \cap \partial B$ is a simple closed curve separating ∂B into two discs. Replace $S \cap B$ with one of these discs, which can be achieved by an ambient isotopy. Performing this operation at each vertex of S results in singularities only of the required form. This proves the claim.

Suppose that the singularities of S occur at the heights $x_1 < \ldots < x_n$. Note that the singularity at x_1 is a birth, and at x_n is a death. We prove the theorem by induction on the number of singularities n. The smallest possible n is two, in which case S is a 2-sphere embedded in the standard way.

Let x_k be the smallest non-birth singularity. If it is a death, then, since S is connected, S is a 2-sphere embedded in the standard way. Hence, we may assume that x_k is a saddle. As x increases to x_k, either

(i) two curves C_1 and C_2 approach to become a single curve C_3, or

(ii) one curve C_4 pinches together form two curves C_5 and C_6.

In (i), we may ambient isotope S, to replace this saddle singularity and the singularities below C_1 and C_2 with a single birth singularity. The theorem then follows by induction.

In (ii), if C_5 and C_6 both lie below death singularities, then S is a 2-sphere ambient isotopic to the standard 2-sphere in S^3.

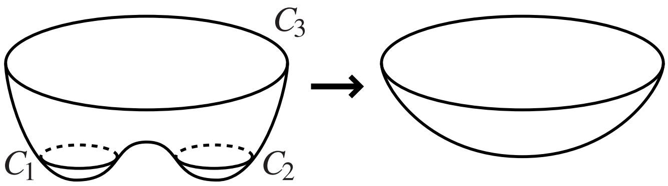


Figure 9.

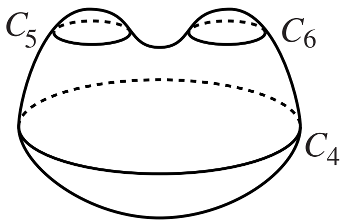


Figure 10.

Suppose therefore that one of these curves (C_5, say) does not lie below a death singularity. The curve C_5 bounds a horizontal disc D. There may be some simple closed curves of $S \cap \mathrm{int}(D)$. But each of these lies above birth singularities. So, we may ambient isotope S, increasing the height of these singularities to above x_k. Hence we may assume that $D \cap S = \partial D = C_5$. By the incompressibility of S, C_5 bounds a disc D' in S. Hence, if we 'compress' S along D, we obtain a surface S' with same genus as S, together with a 2-sphere S^2. Both S^2 and S' have fewer singularities than S. Hence, inductively, S^2 bounds a 3-ball on both sides. One of these 3-balls is disjoint from S'. We may ambient isotope S across this 3-ball onto S'. Thus, S and S' are ambient isotopic. The inductive hypothesis gives that S' (and hence S) is a 2-sphere ambient isotopic to the standard 2-sphere in S^3. $\square$

Using this result, we can prove that any compact 3-manifold M with a single boundary component that is embedded in S^3 is Haken. If S^2 is properly embedded in M, then this 2-sphere separates S^3 into two 3-balls. One of these 3-balls is disjoint from ∂M, and hence lies in M. Therefore M is prime, orientable and compact, and has non-empty boundary. Hence, it is Haken.

Example. Let K be a knot in S^3. We will show in §6 that K has a neighbourhood $\mathcal{N}(K)$ homeomorphic to a solid torus. The 3-manifold $M = S^3 - \mathrm{int}(\mathcal{N}(K))$ is the *exterior* of K. Thus, M is Haken. In fact, it contains an orientable non-separating properly embedded surface, which we now construct.

Pick a planar diagram for the knot K. We view this diagram as lying in $\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$. The knot lies in this plane, except near crossings where one arc skirts above the plane, and one below. Pick an orientation of the knot. Remove each crossing of the diagram in the following way:

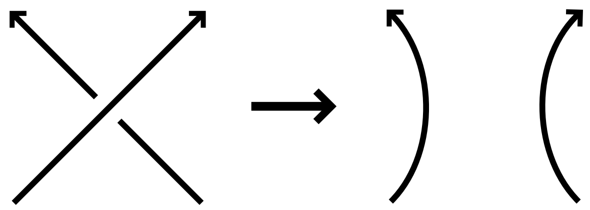


Figure 11.

The result is a collection of simple closed curves in $\mathbb{R}^2$. Attach disjoint discs to these curves, lying above $\mathbb{R}^2$. (Note that the curves may be nested.) Then attach a twisted band at each crossing of K, as in Figure 12. The result is a compact orientable surface S embedded in S^3 with boundary K. Such a surface is known as a *Seifert surface* for K.

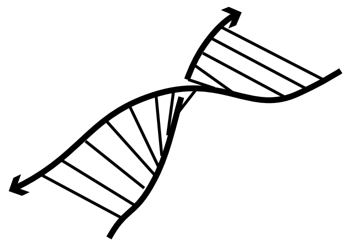


Figure 12.

We may take $\mathcal{N}(K)$ small enough so that $S \cap \mathcal{N}(K)$ is a single annulus. Then $S \cap M$ is an orientable properly embedded surface in M. It is non-separating, since a small loop encircling K intersects the surface transversely in a single point.

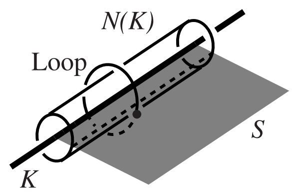


Figure 13.

§4. Basic pl topology

We have already had to state without proof of a number of results of the form 'a certain submanifold has a certain neighbourhood'. It is clear that if we are to argue rigourously, we need to develop a greater understanding of pl topology. The results that we state here without proof can be found in Rourke and Sanderson's book 'Introduction to piecewise-linear topology'.

Regular neighbourhoods

Definition. The *barycentric subdivision* $K^{(1)}$ of the simplicial complex K is constructed as follows. It has precisely one vertex in the interior of each simplex of K (including having a vertex at each vertex of K). A collection of vertices of $K^{(1)}$, in the interior of simplices $\sigma_1, \dots, \sigma_r$ of K, span a simplex of $K^{(1)}$ if and only if σ_1 is a face of σ_2, which is a face of σ_3, etc (possibly after re-ordering $\sigma_1, \dots, \sigma_r$).

An example is given in Figure 14. It is also possible to define $K^{(1)}$ inductively on the dimensions of the simplices of K, as follows. Start with all the vertices of K. Then add a vertex in each 1-simplex of K. Join it to the relevant 0-simplices of K. Then add a vertex in each 2-simplex σ of K. Add 1-simplices and 2-simplices inside σ by 'coning' the subdivision of $\partial\sigma$. Continue analogously with the higher-dimensional simplices.

Definition. The r^{th} *barycentric subdivision* of a simplicial complex K for each $r \in \mathbb{N}$ is defined recursively to be $(K^{(r-1)})^{(1)}$, where $K^{(0)} = K$.

Definition. If L is a subcomplex of the simplicial complex K, then the *regular neighbourhood* $\mathcal{N}(L)$ of K is the closure of the set of simplices in $K^{(2)}$ that intersect L. It is a subcomplex of $K^{(2)}$.

The following result asserts that regular neighbourhoods are essentially independent of the choice of triangulation for K.

Theorem 4.1. (Regular neighbourhoods are ambient isotopic) *Suppose that K' is a subdivision of a simplicial complex K. Let L be a subcomplex of K, and let L' be the subdivision $K' \cap L$. Then the regular neighbourhood of L in K is ambient isotopic to the regular neighbourhood of L' in K'.*

Thus, we may speak of regular neighbourhoods without specifying an initial triangulation.

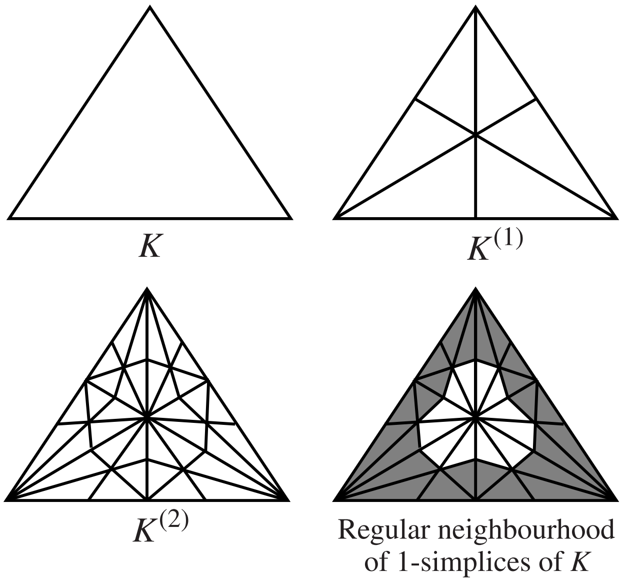


Figure 14.

Handle structures

Definition. A *handle structure* of an n-manifold M is a decomposition of M into $n+1$ sets $\mathcal{H}_0, \dots, \mathcal{H}_n$ having disjoint interiors, such that

- $\mathcal{H}_i$ is a collection of disjoint n-balls, known as *i-handles*, each having a product structure $D^i \times D^{n-i}$,
- for each i-handle $(D^i \times D^{n-i}) \cap (\bigcup_{j=0}^{i-1} \mathcal{H}_j) = \partial D^i \times D^{n-i}$,
- if $H_i = D^i \times D^{n-i}$ (respectively, $H_j = D^j \times D^{n-j}$) is an i-handle (respectively, j-handle) with $j < i$, then $H_i \cap H_j = D^j \times E = F \times D^{n-i}$ for some $(n - j - 1)$-manifold E (respectively, $(i-1)$-manifold F) embedded in ∂D^{n-j} (respectively, ∂D^i).

Here we adopt the convention that D^0 is a single point and $\partial D^0 = \emptyset$.

In words, the third of the above conditions requires that the attaching map of each handle respects the product structures of the handles to which it is attached. For a 3-manifold, this is relevant only for $j = 1$ and $i = 2$.

One should view a handle decomposition as like a CW complex, but with each i-cell thickened to a n-ball.

Theorem 4.2. *Every pl manifold has a handle structure.*

Proof. Pick a triangulation K for the manifold. Let V^i be the vertices of $K^{(1)}$ in the interior of the i-simplices of K. Let $\mathcal{H}^i$ be the closure of the union of the simplices in $K^{(2)}$ touching V^i. These form a handle structure. □

GENERAL POSITION

In $\mathbb{R}^n$ it is well-known that two subspaces, of dimensions p and q, intersect in a subspace of dimension at least $p + q - n$, and that if the dimension of their intersection is more than $p + q - n$, then only a small shift of one of them is required to achieve this minimum. Analogous results hold for subcomplexes of a pl manifold. The *dimension* $\dim(P)$ of a simplicial complex P is the maximal dimension of its simplices.

Proposition 4.3. *Suppose that P and Q are subcomplexes of a closed manifold M, with $\dim(P) = p$, $\dim(Q) = q$ and $\dim(M) = M$. Then there is a homeomorphism $h: M \to M$ isotopic to the identity such that $h(P)$ and Q intersect in a simplicial complex of dimension of at most $p + q - m$.*

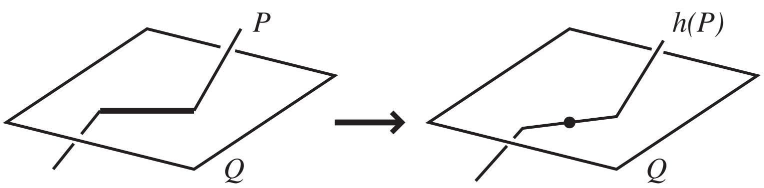


Figure 15.

Then, $h(P)$ and Q are said to be *in general position*. This is one of a number of similar results. They are fairly straightforward, but rather than giving detailed definitions and theorems, we will simply appeal to 'general position' and leave it at that.

Lemma 4.4. *Any pl homeomorphism $\partial D^n \to \partial D^n$ extends to a pl homeomorphism $D^n \to D^n$.*

Proof. See the figure. □

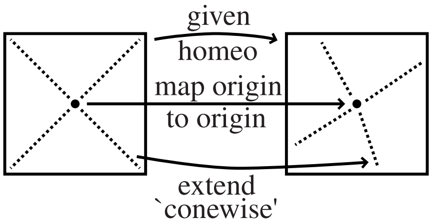


Figure 16.

Remark. The above proof does not extend to the smooth category, and indeed the smooth version is false.

A similar proof gives the following.

Lemma 4.5. *Two homeomorphisms $D^n \to D^n$ which agree on ∂D^n are isotopic.*

Let $r\colon D^n \to D^n$ be the map which changes the sign of the x_n co-ordinate.

Proposition 4.6. *A homeomorphism $D^n \to D^n$ is isotopic either to the identity or to r.*

Proof. By induction on n. First note that there are clearly only two homeomorphisms $\partial D^1 \to \partial D^1$. By Lemma 4.4, these extend to homeomorphisms $D^1 \to D^1$. Now apply Lemma 4.5 to show that any homeomorphism $D^1 \to D^1$ is isotopic to one of these. Now consider a homeomorphism $h\colon \partial D^2 \to \partial D^2$. It takes a 1-simplex σ in ∂D^2 to a 1-simplex in ∂D^2. There are two possibilities up to isotopy for $h|_\sigma$, since σ is a copy of D^1. Note that $\mathrm{cl}(\partial D^2 - \sigma)$ is clearly a copy of a 1-ball. (An explicit homeomorphism is obtained by retracting $\mathrm{cl}(\partial D^2 - \sigma)$ onto one hemisphere of ∂D^2). Hence, each homeomorphism of σ extends to $\partial D^2 - \sigma$, in a way that is unique up to isotopy by Lemmas 4.4 and 4.5. Hence, h is isotopic to $r|_{\partial D^2}$ or $\mathrm{id}|_{\partial D^2}$. Therefore, by Lemma 4.4, any homeomorphism $D^2 \to D^2$ is isotopic to r or id. The inductive step proceeds in all dimensions in this way. □

We end with a couple of further results above spheres and discs that we will use (often implicitly) at a number of points. Their proofs are less trivial than the above results, and are omitted.

Proposition 4.7. *Let $h_1 \colon D^n \to M$ and $h_2 \colon D^n \to M$ be embeddings of the n-ball into an n-manifold. Then there is a homeomorphism $h \colon M \to M$ isotopic to the identity such that $h \circ h_1$ is either h_2 or $h_2 \circ r$.*

Proposition 4.8. *The space obtained by gluing two n-balls along two closed $(n-1)$-balls in their boundaries is homeomorphic to an n-ball.*

§5. Constructing 3-manifolds

The aim now is to give some concrete constructions of 3-manifolds. This will be a useful application of the pl theory outlined in the last section.

Construction 1. Heegaard splittings.

Definition. A *handlebody of genus g* is the 3-manifold with boundary obtained from a 3-ball B^3 by gluing $2g$ disjoint closed 2-discs in ∂B^3 in pairs via orientation-reversing homeomorphisms.

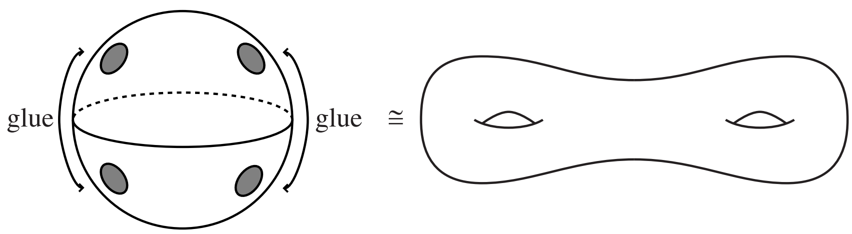


Figure 17.

Lemma 5.1. *Let H be a connected orientable 3-manifold with a handle structure consisting of only 0-handles and 1-handles. Then H is a handlebody.*

Proof. Pick an ordering on the handles of H, and reconstruct H by regluing these balls, one at a time, as specified by this ordering. At each stage, we identify discs, either in distinct components of the 3-manifold, or in the same component of the 3-manifold. Perform all of the former identifications first. The result is a 3-ball. Then perform all of the latter identifications. Each must be orientation-reversing, since H is orientable. Hence, H is a handlebody. □

Let H_1 and H_2 be two genus g handlebodies. Then we can construct a 3-manifold M by gluing H_1 and H_2 via a homeomorphism $h\colon \partial H_1 \to \partial H_2$. This is known as a *Heegaard splitting* of M.

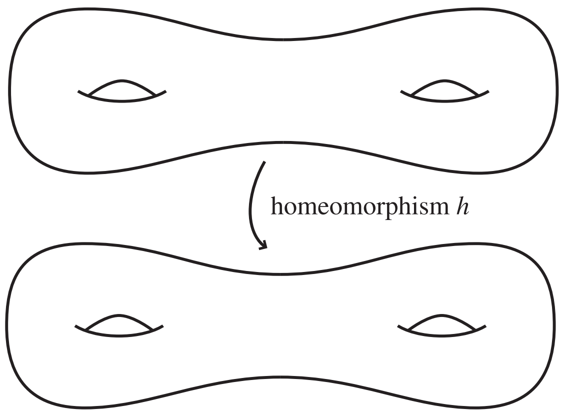


Figure 18.

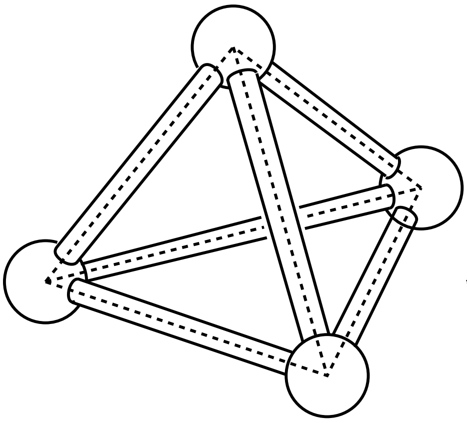
Exercise. Take two copies of the same genus g handlebody and glue their boundaries via the identity homeomorphism. Show that the resulting space is homeomorphic to the connected sum of g copies of $S^1 \times S^2$.

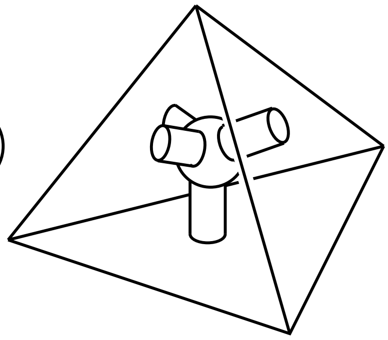
Exercise. Show that, if H is the genus g handlebody embedded in S^3 in the standard way, then $S^3 - \mathrm{int}(H)$ is also a handlebody. Hence, show that S^3 has Heegaard splittings of all possible genera.

Example. A common example is the case where two solid tori are glued along their boundaries. By the above two exercises, S^3 and $S^2 \times S^1$ have such Heegaard splittings. However, other manifolds can be constructed in this way. A *lens space* is a 3-manifold with a genus 1 Heegaard splitting which is not homeomorphic to S^3 or $S^2 \times S^1$. Note that there are many ways to glue the two solid tori together, because there are many possible homeomorphisms from a torus to itself, constructed as follows. View T^2 as $\mathbb{R}^2/\sim$, where $(x, y) \sim (x+1, y)$ and $(x, y) \sim (x, y+1)$. Then any linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with integer matrix entries and determinant ± 1 descends to a homeomorphism $T^2 \to T^2$.

Theorem 5.2. *Any closed orientable 3-manifold M has a Heegaard splitting.*

Proof. Pick a handle structure for M. The 0-handle and 1-handles form a handlebody. Similarly, the 2-handles and 3-handles form a handlebody. (If one views each i-handle in a handle structure for a closed n-manifold as an $(n-i)$-handle, the result is again a handle structure.) $\square$





0-handles and 1-handles   2-handles and 3-handles

Figure 19.

Construction 2. The mapping cylinder.

Start with a compact orientable surface F. Now glue the two boundary components of $F \times [0,1]$ via an orientation-reversing homeomorphism $h\colon F \times \{0\} \to F \times \{1\}$. The result is a compact orientable 3-manifold $(F \times [0,1])/h$ known as the *mapping cylinder for* h.

Exercise. If two homeomorphisms h_0 and h_1 are isotopic then $(F \times [0,1])/h_0$ and $(F \times [0,1])/h_1$ are homeomorphic.

However, there are many homeomorphisms $F \to F$ not isotopic to the identity.

Definition. Let C be a simple closed curve in the interior of the surface F. Let $\mathcal{N}(C) \cong S^1 \times [-1,1]$ be a regular neighbourhood of C. Then a *Dehn twist* about C is the map $h\colon F \to F$ which is the identity outside $\mathcal{N}(C)$, and inside $\mathcal{N}(C)$ sends (θ, t) to $(\theta + \pi(t+1), t)$.

Note. The choice of identification $\mathcal{N}(C) \cong S^1 \times [-1,1]$ affects the resulting homeomorphism, since it is possible to twist in 'both directions'.

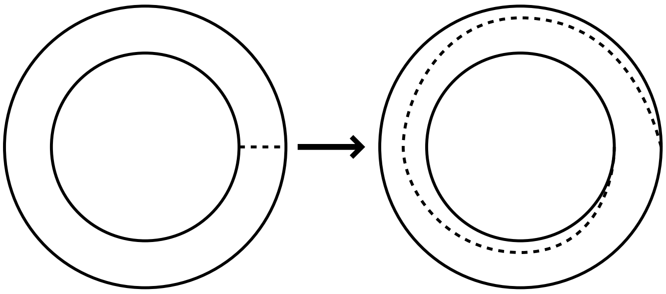


Figure 20.

Exercise. If C bounds a disc in F or is parallel to a boundary component, then a Dehn twist about C is isotopic to the identity. But it is in fact possible to show that if neither of these conditions holds, then a Dehn twist about C is never isotopic to the identity.

Theorem 5.3. [Dehn, Lickorish] *Any orientation preserving homeomorphism of a compact orientable surface to itself is isotopic to the composition of a finite number of Dehn twists.*

CONSTRUCTION 3. Surgery

Let L be a link in S^3 with n components. Then $\mathcal{N}(L)$ is a collection of solid tori. Let M be the 3-manifold obtained from $S^3 - \mathrm{int}(\mathcal{N}(L))$ by gluing in n solid tori $\bigcup_{i=1}^n S^1 \times D^2$, via a homeomorphism $\partial(\bigcup_{i=1}^n S^1 \times D^2) \to \partial\mathcal{N}(L)$. The resulting 3-manifold is *obtained by surgery along L*.

There are many possible ways of gluing in the solid tori, since there are many homeomorphisms from a torus to itself.

Theorem 5.4. [Lickorish, Wallace] *Every closed orientable 3-manifold M is obtained by surgery along some link in S^3.*

Proof. Let $H_1 \cup H_2$ be a Heegaard splitting for M, with gluing homeomorphism $f\colon \partial H_1 \to \partial H_2$. Let $g\colon \partial H_1 \to \partial H_2$ be a gluing homeomorphism for a Heegaard splitting of S^3 of the same genus. Note that H_1 and H_2 inherit orientations from M and S^3, and, with respect to these orientations, f and g are orientation reversing. Then, by Theorem 5.3, $g^{-1} \circ f$ is isotopic to a composition of Dehn twists, $\tau_1, \dots, \tau_n$ along curves $C_1, \dots, C_n$, say. Let $k\colon \partial H_1 \times [n, n+1] \to \partial H_1 \times [n, n+1]$ be the isotopy between $\tau_n \circ \dots \circ \tau_1$ and $g^{-1} \circ f$. A regular neighbourhood

$\mathcal{N}(\partial H_1)$ of ∂H_1 in H_1 is homeomorphic to a product $\partial H_1 \times [0, n+1]$, say, with $\partial H_1 \times \{n+1\} = \partial H_1$. (See Theorem 6.1 in the next section.) For $i = 1, \ldots, n$, let $L_i = \tau_1^{-1} \ldots \tau_{i-1}^{-1} C_i \times \{i - 3/4\} \subset H_1 \subset M$. Define a homeomorphism

$$M - \bigcup_{i=1}^{n} \mathrm{int}(\mathcal{N}(L_i)) \to S^3 - \mathrm{int}(\mathcal{N}(L))$$

$$\begin{aligned}
H_1 - (\partial H_1 \times [0, n+1]) &\xrightarrow{\mathrm{id}} H_1 - (\partial H_1 \times [0, n+1]) \\
(\partial H_1 - \mathcal{N}(C_1)) \times [0, 1/2] &\xrightarrow{\mathrm{id}} (\partial H_1 - \mathcal{N}(C_1)) \times [0, 1/2] \\
\partial H_1 \times [1/2, 1] &\xrightarrow{\tau_1} \partial H_1 \times [1/2, 1] \\
(\partial H_1 - \mathcal{N}(\tau_1^{-1} C_2)) \times [1, 3/2] &\xrightarrow{\tau_1} (\partial H_1 - \mathcal{N}(C_2)) \times [1, 3/2] \\
\partial H_1 \times [3/2, 2] &\xrightarrow{\tau_2 \tau_1} \partial H_1 \times [3/2, 2] \\
&\ldots \\
\partial H_1 \times [n - 1/2, n] &\xrightarrow{\tau_n \ldots \tau_1} \partial H_1 \times [n - 1/2, n] \\
\partial H_1 \times [n, n+1] &\xrightarrow{k} \partial H_1 \times [n, n+1] \\
H_2 &\xrightarrow{\mathrm{id}} H_2
\end{aligned}$$

Here, L is a collection of simple closed curves in $H_1 \subset S^3$. These homeomorphisms all agree, since $\tau_i \ldots \tau_1$ and $\tau_{i-1} \ldots \tau_1$ agree on $\partial H_1 - \tau_1^{-1} \ldots \tau_{i-1}^{-1} \mathcal{N}(C_i)$. Therefore, M is obtained from S^3 by first removing a regular neighbourhood of the link L, and then gluing in n solid tori. □

§6. Regular neighbourhoods and fibre bundles

Theorem 6.1. (Regular neighbourhoods of submanifolds) *Let L be a submanifold of pl manifold M. Then $\mathcal{N}(L)$ is the total space of a fibre bundle over L, with fibre a disc D^n, and with the inclusion $L \to \mathcal{N}(L)$ being a section.*

In this course, we will only consider very simple bundles. We therefore only give the briefest outline of their theory. Normally bundles are dealt with in the smooth category, but there is of course a pl version. This is less satisfactory in high dimensions, but in dimension three, it works well.

Definition. A map $p: B \to M$ is a *fibre bundle over M with total space B and fibre F* (or an *F-bundle*) if M has an open cover $\{U_\alpha\}$ such that

- the closure $\overline{U}_\alpha$ of each U_α is simplicial, and
- each $p^{-1}(\overline{U}_\alpha)$ is (pl) homeomorphic to $F \times \overline{U}_\alpha$ so that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(\overline{U}_\alpha) & \xrightarrow{\cong} & F \times \overline{U}_\alpha \\ \Big\downarrow p & & \Big\downarrow \text{projection onto 2nd factor} \\ \overline{U}_\alpha & = & \overline{U}_\alpha \end{array}$$

If U_α and U_β intersect, then there are two maps

$$p^{-1}(\overline{U}_\alpha \cap \overline{U}_\beta) \to F \times (\overline{U}_\alpha \cap \overline{U}_\beta),$$

one given via U_α, one via U_β. Hence, we obtain a map $g_{\beta\alpha}: F \times (\overline{U}_\alpha \cap \overline{U}_\beta) \to F \times (\overline{U}_\alpha \cap \overline{U}_\beta)$, such $g_{\beta\alpha}|_{F\times\{x\}}$ is a homeomorphism onto $F \times \{x\}$ for each $x \in \overline{U}_\alpha \cap \overline{U}_\beta$. These maps $g_{\beta\alpha}$ are known as the *transition maps*, and satisfy the following conditions:

1. $g_{\alpha\alpha} = \mathrm{id}$,
2. $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$,
3. $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$.

Usually, one insists that, for each α and β and each $x \in \overline{U}_\alpha \cap \overline{U}_\beta$, $g_{\alpha\beta}|_{F\times\{x\}}$ should lie in some specified subgroup of $\mathrm{Homeo}(F, F)$, known as the *structure group* of the bundle. In this case, all we insist is that these homeomorphisms be pl.

Note that a fibre bundle over M with fibre F can be specified by an open cover $\{U_\alpha\}$ of M (with each $\overline{U}_\alpha$ simplicial), together with transition maps satisfying the above three conditions.

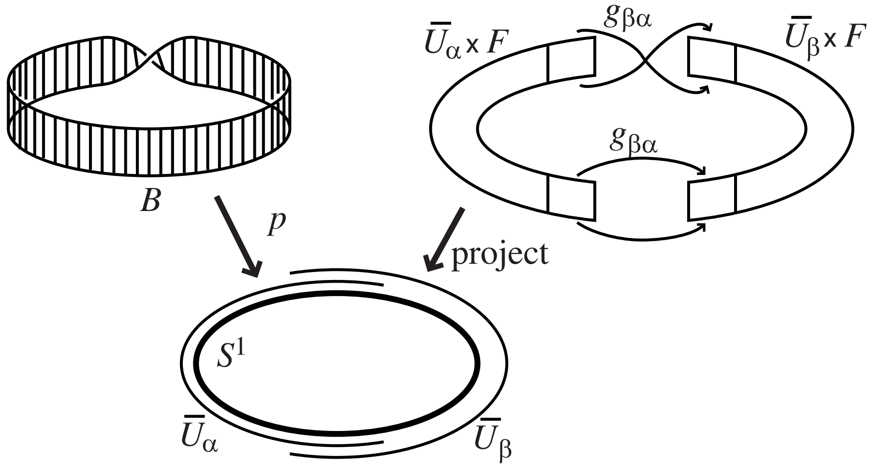


Figure 21.

Definition. A *section* of a fibre bundle $p: B \to M$ is a map $s: M \to B$ such that $p \circ s = \mathrm{id}_M$.

Sketch proof of Theorem 6.1. Pick a triangulation of M in which L is simplicial. This induces handle structures on L and M. Each i-handle of L is contained in an i-handle of M. The union of these handles of M containing L forms $\mathcal{N}(L)$. Careful choice of product structures on the handles (starting with the highest index handles and working downwards) can be used to define the bundle map $p: \mathcal{N}(L) \to L$. Each U_α is (a small extension) of a handle of L. □

Definition. Two bundles $p_1: B_1 \to M$ and $p_2: B_2 \to M$ are *equivalent* if there is a homeomorphism $h: B_1 \to B_2$ so that the following commutes:

$$\begin{array}{ccc} B_1 & \xrightarrow{h} & B_2 \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_2} \\ M & = & M \end{array}$$

Definition. If $p: B \to M$ is a fibre bundle and $f: M' \to M$ is any map, then there is a bundle over M', known as the *pull-back bundle*. It is constructed by taking the open cover $\{U_\alpha\}$ via which M is defined, and letting $\{f^{-1}(U_\alpha)\}$ be the open cover for M'. If $g_{\alpha\beta}: F \times (\overline{U}_\alpha \cap \overline{U}_\beta) \to F \times (\overline{U}_\alpha \cap \overline{U}_\beta)$ is a transition map then the transition map at a point x of $f^{-1}(\overline{U}_\alpha) \cap f^{-1}(\overline{U}_\beta)$ is given by $g_{\alpha\beta}|_{F \times \{f(x)\}}$.

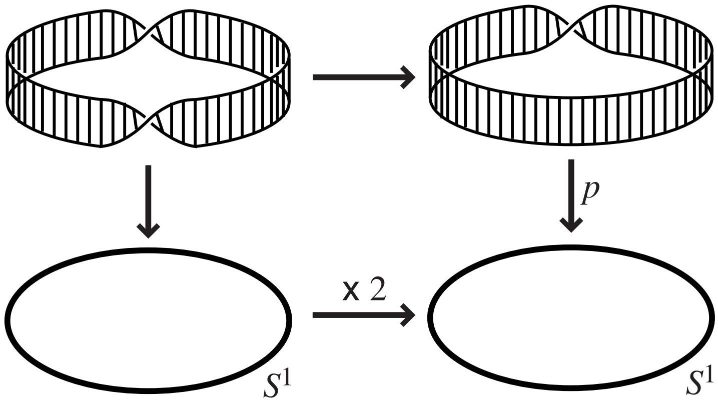


Figure 22.

Examples. Let B be any bundle over M. If $i: M' \to M$ is an inclusion map, then the pull-back bundle is the restriction of the bundle to M'. The pull-back of B with respect to id_M is the same bundle as B. The pull-back with respect to a constant map $M' \to M$ is a product bundle.

The following important result is not very difficult. Its proof can be found in Husemoller's book 'Fibre Bundles'.

Theorem 6.2. *Let M be compact, and let $p: B \to M \times [0,1]$ be a fibre-bundle. Then the associated bundles over $M \times \{0\}$ and $M \times \{1\}$ are equivalent.*

Corollary 6.3. *A bundle over a contractible space M is a product bundle.*

Proof. Since M is contractible, there is a homotopy $M \times [0,1] \to M$ between id_M and a constant map. Pull back the bundle over M to a bundle over $M \times [0,1]$. The bundle over $M \times \{0\}$ is the original bundle. The bundle over $M \times \{1\}$ is the product bundle. They are equivalent by Theorem 6.2. $\square$.

Lemma 6.4. *For each $n \in \mathbb{N}$, there are precisely two D^n-bundles over S^1 up to bundle equivalence.*

Proof. The two D^n-bundles over S^1 are constructed as follows. Start with the product bundle $D^n \times [0,1]$ over $[0,1]$, and glue $D^n \times \{0\}$ to $D^n \times \{1\}$ via some homeomorphism. The result is a D^n-bundle over S^1. It is easy to see that isotopic gluing homeomorphisms give equivalent bundles. By Proposition 4.6, there are two isotopy classes of such homeomorphisms. To see that the bundles are inequivalent, note that their underlying spaces are not homeomorphic: one is orientable and one is not.

Now we must show that every D^n-bundle over S^1 is equivalent to one of these. Pick a point $x \in S^1$. Then, restricting to the bundle over $S^1 - \mathrm{int}(\mathcal{N}(x))$ is a bundle over the interval, which by Corollary 6.3 is a product. Hence, our bundle is constructed as above. □

We now give a characterisation of whether a manifold is orientable.

Proposition 6.5. *An n-manifold M is orientable if and only if it contains no embedded copy of the total space of the non-orientable D^{n-1}-bundle over S^1.*

Proof. If such a bundle embeds in M, then some triangulation of M is non-orientable, and hence M is non-orientable.

Conversely, suppose that M contains no such bundle. Pick an orientation on some n-simplex of M. This specifies unique compatible orientations on its neighbouring n-simplices. Repeat with these simplices. In this way, we orient M, unless at some stage we return to an n-simplex and assign it an orientation the opposite from its original orientation. This specifies a loop, running between the n-simplices through the $(n-1)$-dimensional faces. We may take this loop ℓ to be embedded. Then $\mathcal{N}(\ell)$ is the required non-orientable D^{n-1}-bundle over S^1. □

The total space of the non-orientable D^{n-1}-bundle over S^1 is the Möbius band for $n = 2$ and the solid Klein bottle for $n = 3$.

Proposition 6.6. *Let S be a surface properly embedded in a compact orientable 3-manifold M. Then S is orientable if and only if $\mathcal{N}(S)$ is homeomorphic to $S \times I$.*

Proof. It suffices to consider the case where S is connected. Let $p\colon \mathcal{N}(S) \to S$ be the I-bundle over S from Theorem 6.1. Suppose first that S has non-empty boundary. Then there is a collection A of disjoint properly embedded arcs in S, such that cutting S along A gives a disc D. Then, by Corollary 6.3, the restriction of p to $p^{-1}(D)$ is a product I-bundle. Now identify arcs in ∂D in pairs to give S. These arcs inherit an orientation from some orientation on ∂D.

If two arcs α_1 and α_2 are glued so that their orientations agree, then S contains an embedded Möbius band and so is non-orientable. When $\alpha_1 \times I$ is glued to $\alpha_2 \times I$, the orientations of the I factors must be reversed (otherwise M would contain a solid Klein bottle and hence be non-orientable). Hence, the ∂I-bundle over S is connected, and therefore $\mathcal{N}(S)$ is not homeomorphic to $S \times I$.

Suppose therefore that each pair of arcs α_1 and α_2 in ∂D are identified in a way that reverses orientation. Then S is orientable. Also, the gluing map between $\alpha_1 \times I$ and $\alpha_2 \times I$ preserves the orientation of the I-factor, otherwise M would contain a solid Klein bottle. Hence, after an isotopy of the gluing maps, we may assume that it is the identity in the I-factors. Hence, $\mathcal{N}(S)$ is a product I-bundle.

Now consider the case where S is closed. Remove the interior of a small disc D to give a surface S'. Then S is orientable if and only if S' is. If $\mathcal{N}(S)$ is product I-bundle, then its restriction to S' is. Conversely, if its restriction to S' is a product I-bundle, then we may extend the product structure over $p^{-1}(D)$ to give a product structure on $\mathcal{N}(S)$. □

A codimension one submanifold X of a manifold is known as *two-sided* if $\mathcal{N}(X)$ is a product I-bundle. The existence of a product neighbourhood for a properly embedded orientable surface S in an orientable 3-manifold M is very important. For example, it is vital in the proof of Theorem 3.3, which asserts that S is incompressible if and only if it is π_1-injective. This can in fact fail for non-orientable surfaces. For example, there is a non-orientable incompressible embedded surface in some lens space which is not π_1-injective.

§7. Homology of 3-manifolds

Definition. For $i \in \mathbb{Z}_{\geq 0}$, the i^{th} *Betti number* $\beta_i(M)$ of a space M is the dimension of $H_i(M;\mathbb{Q})$ viewed as a vector space over $\mathbb{Q}$.

Definition. The *Euler characteristic* $\chi(M)$ of a compact triangulable space M is

$$\sum_i (-1)^i \beta_i(M).$$

Theorem 7.1. *Pick any triangulation of a compact space M, and let σ_i be the number of i-simplices in this triangulation. Then $\chi(M) = \sum_i (-1)^i \sigma_i$.*

Remark. If $H_i(M) \cong \mathbb{Z}^a \oplus T$, where each element of T has finite order, then $\beta_i(M) = a$.

The following result, which we quote without proof, is one of the cornerstones of manifold theory.

Theorem 7.2. (Poincaré duality) *Let M be a compact connected orientable n-manifold. Then for each i, $H_i(M, \partial M; \mathbb{Q}) \cong H_{n-i}(M; \mathbb{Q})$.*

Remark. The corresponding statements for coefficients in $\mathbb{Z}$ is not true.

Corollary 7.3. *Let M be a closed orientable m-manifold, with m odd. Then $\chi(M) = 0$.*

Corollary 7.4. *For a compact orientable m-manifold M, with m odd, $\chi(M) = (1/2)\chi(\partial M)$.*

Proof. Let DM be two copies of M glued along ∂M, via the 'identity' map. Then a triangulation of M induces one for DM. Counting i-simplices gives $0 = \chi(DM) = 2\chi(M) - \chi(\partial M)$. □

Theorem 7.5. *Let M be a compact orientable 3-manifold, with at least one component of ∂M not a 2-sphere. Then there is an element of $H_1(\partial M)$ which has infinite order in $H_1(M)$.*

Proof. Let $\hat{M}$ be the 3-manifold obtained by attaching a 3-ball to each 2-sphere component of ∂M. Then $H_1(\hat{M}) \cong H_1(M)$. Since $\hat{M}$ is not closed, $H_3(\hat{M}) = 0$ and so $\beta_3(\hat{M}) = 0$. Since M is orientable, so is $\hat{M}$ and $\partial\hat{M}$. Since $\partial\hat{M}$ contains no 2-spheres, $\chi(\partial\hat{M}) \leq 0$. Corollary 7.4 implies that $\chi(\hat{M}) \leq 0$. But $\chi(\hat{M}) = \beta_0(\hat{M}) - \beta_1(\hat{M}) + \beta_2(\hat{M}) - \beta_3(\hat{M}) = 1 - \beta_1(\hat{M}) + \beta_2(\hat{M}) \leq 0$. So, $\beta_1(\hat{M}) > \beta_2(\hat{M})$. Therefore, in the long exact sequence of the pair $(\hat{M}, \partial\hat{M})$, the map $H_1(\hat{M}; \mathbb{Q}) \to H_1(\hat{M}, \partial\hat{M}; \mathbb{Q})$ has non-trivial kernel. Hence, there is an element of $H_1(\hat{M}; \mathbb{Q})$ in the image of $H_1(\partial\hat{M}; \mathbb{Q})$. Clearing denominators from the coefficients gives an infinite order element of $H_1(\hat{M})$ in the image of $H_1(\partial\hat{M})$. The following diagram commutes, where each map is induced by inclusion.

$$\begin{array}{ccc} H_1(\partial \hat{M}) & \longrightarrow & H_1(\hat{M}) \\ \downarrow \cong & & \uparrow \cong \\ H_1(\partial M) & \rightarrow & H_1(M) \end{array}$$

This proves the theorem. □

We introduce some standard terminology.

Definition. A 3-manifold M is *irreducible* if any embedded 2-sphere bounds a 3-ball in M.

By Proposition 3.5, a 3-manifold is irreducible if and only if it is prime and not $S^2 \times S^1$.

Theorem 7.6. *Let M be a compact irreducible 3-manifold with $H_1(M)$ infinite. Then M contains a connected 2-sided non-separating properly embedded incompressible surface S, which is not a 2-sphere. Furthermore, if there is an infinite order element of $H_1(M)$ in the image of $H_1(\partial M)$, then we may guarantee that ∂S has non-zero signed intersection number with some loop in ∂M.*

Lemma 7.7. *Let M be a compact connected 3-manifold and let X be a space with $\pi_2(X) = 0$. Then, for any basepoints $m \in M$ and $x \in X$, any homomorphism $\pi_1(M, m) \to \pi_1(X, x)$ is induced by a map $M \to X$.*

Proof. Pick a triangulation of M with m a 0-simplex. The 0-simplices and 1-simplices form a graph in M. Pick a maximal tree T in this graph and map it to x. For each remaining 1-simplex σ_1 of M, there is a unique path in T joining the endpoints of σ_1. The union of this path with σ_1 forms a loop which (when oriented) represents an element of $\pi_1(M, m)$. The given homomorphism $\pi_1(M, m) \to \pi_1(X, x)$ determines a loop in X (up to homotopy). Send σ_1 to this loop.

Let σ_2 be any 2-simplex of M. Its three boundary 1-simplices $\partial\sigma_2$ have been mapped into X. Since $\partial\sigma_2$ is homotopically trivial in M and group homomorphisms send the identity element to the identity element, the image of $\partial\sigma_2$ is homotopically trivial in X. Using this homotopy, we may extend our map over σ_2.

Now, let σ_3 be any 3-simplex of M. We have mapped $\partial\sigma_3$ to a 2-sphere in

X. Since $\pi_2(X) = 0$, this extends to a map of the 3-ball into X. Hence, we may extend over each 3-simplex. □

Lemma 7.8. *Let M be a compact irreducible 3-manifold, and let X be a pl k-manifold containing a properly embedded 2-sided $(k-1)$-submanifold Y. Suppose that $\ker(\pi_1(Y) \to \pi_1(X)) = 1$ and $\pi_2(Y) = \pi_2(X - Y) = \pi_3(X) = 0$. Then any map $f\colon M \to X$ is homotopic to a map g such that*

(i) each component of $g^{-1}(Y)$ is a properly embedded 2-sided incompressible surface in M,

(ii) no component of $g^{-1}(Y)$ is a 2-sphere, and

(iii) for properly chosen product neighbourhoods $\mathcal{N}(Y)$ and $\mathcal{N}(g^{-1}(Y))$, the map $g|_{\mathcal{N}(g^{-1}(Y))}$ sends fibres homeomorphically onto fibres.

Proof. Since Y is a pl submanifold of X, there is a triangulation of X in which Y is a union of simplices. By assumption, $\mathcal{N}(Y)$ is a product I-bundle. Hence, we may alter the triangulation of X, by replacing each simplex σ of Y with the standard triangulation of the product $\sigma \times [-1, 1]$. Then $Y = Y \times \{0\}$ embeds in X transversely to the triangulation. Using the Simplicial Approximation Theorem, we may subdivide a given triangulation of M and perform a homotopy to f so that afterwards it is simplicial.

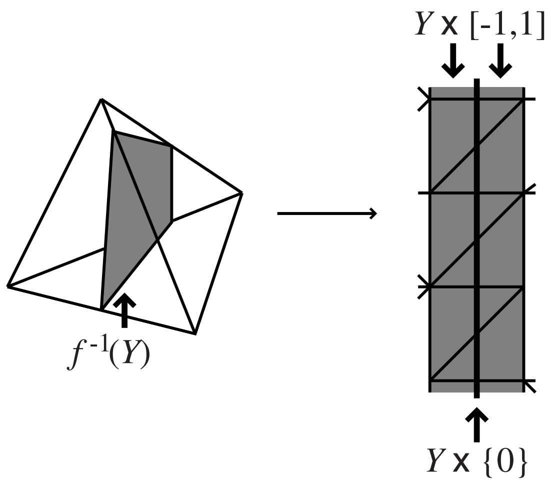


Figure 23.

Then each component of $f^{-1}(Y)$ is a properly embedded 2-sided surface, satisfying condition (iii) relative to $Y \times [-1/2, 1/2]$ and $f^{-1}(Y \times [-1/2, 1/2])$. If $f^{-1}(Y)$ is incompressible, and no component is a 2-sphere, we are done.

Suppose now that D is a compressing disc for $f^{-1}(Y)$. Choose a regular neighbourhood $\mathcal{N}(D)$ in M such that $A = \mathcal{N}(D) \cap f^{-1}(Y)$ is an annulus properly embedded in $\mathcal{N}(D)$. Let D_1 and D_2 be disjoint discs properly embedded in $\mathcal{N}(D)$ such that $\partial D_1 \cup \partial D_2 = \partial A$. Define $f_1 \colon M \to X$ as follows. Put $f_1|_{M-\mathrm{int}(\mathcal{N}(D))} = f|_{M-\mathrm{int}(\mathcal{N}(D))}$. The map $f|_{D_i}$ is a trivialising homotopy for the curve $f|_{\partial D_i}$. Since $\ker(\pi_1(Y) \to \pi_1(X)) = 1$, we may extend $f_1|_{\partial D_i}$ to a map $f_1|_{D_i}$ into Y. Extend f_1 over a small neighbourhood $\mathcal{N}(D_i)$ of D_i using the product structure of $\mathcal{N}(Y)$. Then $\mathcal{N}(D) - (\mathrm{int}(\mathcal{N}(D_1 \cup D_2))$ is three 3-balls. On their boundaries, f_1 is already defined, mapping into $Y - X$. Since $\pi_2(Y - X) = 0$, we may extend f_1 over all of $\mathcal{N}(D)$, avoiding Y. Then $f_1^{-1}(Y) = f^{-1}(Y) \cup D_1 \cup D_2 - \mathrm{int}(A)$. Thus, $f_1^{-1}(Y)$ is obtained from $f^{-1}(Y)$ via a compression. It therefore reduces the complexity of the surface, defined in §3. Note that f and f_1 differ only within a 3-ball, and therefore they are homotopic, since $\pi_3(X) = 0$.

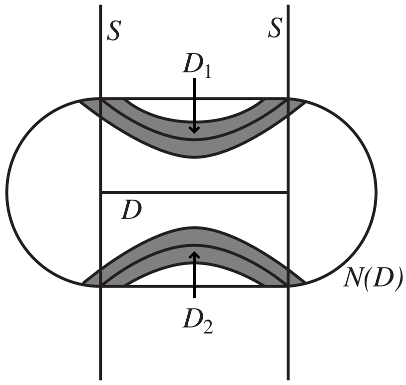


Figure 24.

If some component of $f^{-1}(Y)$ is a 2-sphere, then it bounds a 3-ball B in M. We define a map $f_1 \colon M \to X$ as follows. Let $f|_{M-\mathrm{int}(B)} = f_1|_{M-\mathrm{int}(B)}$. Using that $\pi_2(Y) = 0$, we may extend $f|_B$ to a map $f_1|_B \colon B \to Y$. Then use the product structure on $\mathcal{N}(Y)$ to define a small homotopy so that $f_1(B) \cap Y = \emptyset$, removing the 2-sphere component of $f^{-1}(Y)$. This leaves the complexity of the surface unchanged, but it reduces the number of components. Hence, we eventually obtained the map g as required. □

Proof of Theorem 7.6. Since $H_1(M)$ is infinite but finitely generated, it has $\mathbb{Z}$ as a summand. Hence, there is a surjective homomorphism $H_1(M) \to \mathbb{Z}$. If there is an infinite order element of $H_1(M)$ in the image of $H_1(\partial M)$, we may assume that the composition $H_1(\partial M) \to H_1(M) \to \mathbb{Z}$ is surjective.

Now, there is a surjective homomorphism $\pi_1(M) \to H_1(M)$ which sends a based oriented loop in M to a sum of oriented 1-simplices representing that loop. Hence, there is a surjection $\pi_1(M) \to \mathbb{Z}$. In the case where there is an infinite order element of $H_1(M)$ in the image of $H_1(\partial M)$, we may take $\pi_1(\partial M) \to \pi_1(M) \to \mathbb{Z}$ to be surjective. The map $\pi_1(M) \to \mathbb{Z}$ is induced by a map $M \to S^1$, by Lemma 7.7. Apply Lemma 7.8 to a point Y in S^1. Then some component of $g^{-1}(Y)$ is a 2-sided non-separating incompressible surface S in M that is not a 2-sphere. If $\pi_1(\partial M) \to \pi_1(M) \to \mathbb{Z}$ is surjective, a loop in ∂M mapping to $1 \in \mathbb{Z}$ must have odd signed intersection number with ∂S. □

§8. Hierarchies

In this section, we consider not just a single incompressible surface, but a whole sequence of them.

Terminology. Let M be a 3-manifold, containing an incompressible surface S. Then $M_S = M - \mathrm{int}(\mathcal{N}(S))$ is the result of *cutting M along S*.

Definition. A *partial hierarchy* for a Haken 3-manifold M_1 is a sequence of 3-manifolds $M_1, \dots, M_n$, where M_{i+1} is obtained from M_i by cutting along an orientable incompressible properly embedded surface in M_i, no component of which is a 2-sphere. This is a *hierarchy* if, in addition, M_n is a collection of 3-balls. We denote (partial) hierarchies as follows:

$$M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n.$$

Example. The following is a hierarchy for $S^1 \times S^1 \times S^1$:

$$S^1 \times S^1 \times S^1 \xrightarrow{S^1 \times S^1 \times \{*\}} S^1 \times S^1 \times I \xrightarrow{S^1 \times \{*\} \times I} S^1 \times I \times I \xrightarrow{\{*\} \times I \times I} I \times I \times I.$$

Example. An example of hierarchy for a knot exterior is given below.

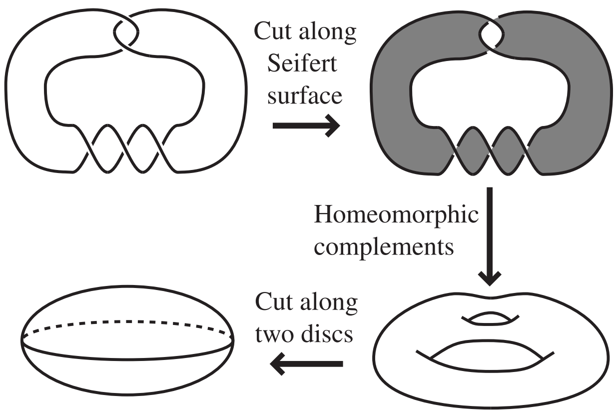


Figure 25.

Non-example. Let M be any 3-manifold with non-empty boundary. Let D be a disc in ∂M. Let D' be D with its interior pushed a little into the interior of M. Then decomposing M along D' gives a copy of M and a 3-ball. Hence, we may repeat this process indefinitely.

Non-example. Let S be the genus one orientable surface with one boundary component. Then $S \times I$ is homeomorphic to a genus two handlebody. Pick a simple closed non-separating curve C in the interior of S. Then $C \times I$ is a properly embedded annulus that is π_1-injective and hence incompressible. Cutting S along C gives a pair of pants $F_{0,3}$, and $F_{0,3} \times I$ is again a genus two handlebody. Hence, we may cut along a similar surface again, and repeat indefinitely.

Lemma 8.1. *Let M be a compact orientable irreducible 3-manifold. Let S be a properly embedded incompressible surface, no component of which is a 2-sphere. Then M_S is irreducible, and hence Haken since $\partial M_S \neq \emptyset$.*

Proof. Let S^2 be a 2-sphere in M_S. As M is irreducible, it bounds a 3-ball in M. If this 3-ball contained any component of S, then S would be compressible, by Theorem 3.8. Hence, S is disjoint from the 3-ball, and so the 3-ball lies in M_S. $\square$

Despite the 'non-examples' above, the following theorem is in fact true.

Theorem 8.2. *Every Haken 3-manifold has a hierarchy.*

Theorem 8.2 will be proved in §11, but first, we show why hierarchies are useful.

9. Boundary patterns and the Loop Theorem

Definition. A *boundary pattern* P in a 3-manifold M is a (possibly empty) collection of disjoint simple closed curves and trivalent graphs in ∂M, such that no simple closed curve in ∂M intersects P transversely in a single point.

If S is a 2-sided surface properly embedded in a compact 3-manifold M, with ∂S intersecting P transversely (and missing the vertices of P), then the manifold M_S obtained by cutting along S inherits a boundary pattern, as follows. Note that ∂M_S is the union of subsurfaces, one of which is $\partial M \cap \partial M_S$, the other of which is $\partial \mathcal{N}(S) \cap \partial M_S$, which is two copies S_1 and S_2 of S. Then, M_S inherits a boundary pattern $(P \cap \partial M_S) \cup \partial S_1 \cup \partial S_2$.

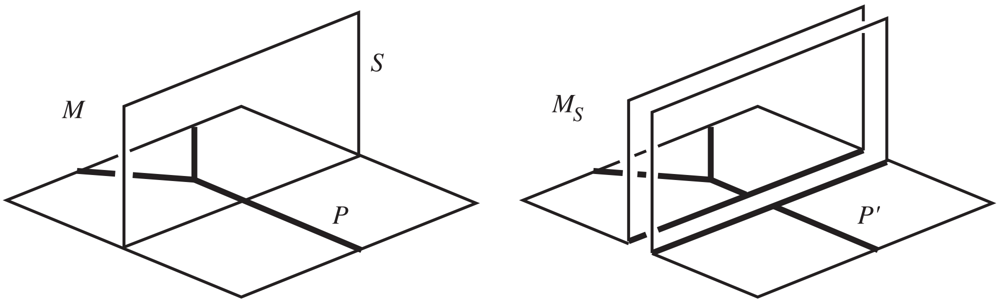


Figure 26.

The motivation for defining boundary patterns is as follows. If

$$M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n$$

is a partial hierarchy for a 3-manifold M_1, then ∂M_n is a union of subsurfaces, which come from bits of ∂M_1 and $S_1, \dots, S_{n-1}$. The union of the boundaries of these bits of surface forms a boundary pattern for M_n.

Definition. A boundary pattern P for M is *essential* if, for each disc D properly embedded in M with $\partial D \cap P$ at most three points, there is a disc $D' \subset \partial M$ with $\partial D' = \partial D$, and D' containing at most one vertex of P and no simple closed curves of P.

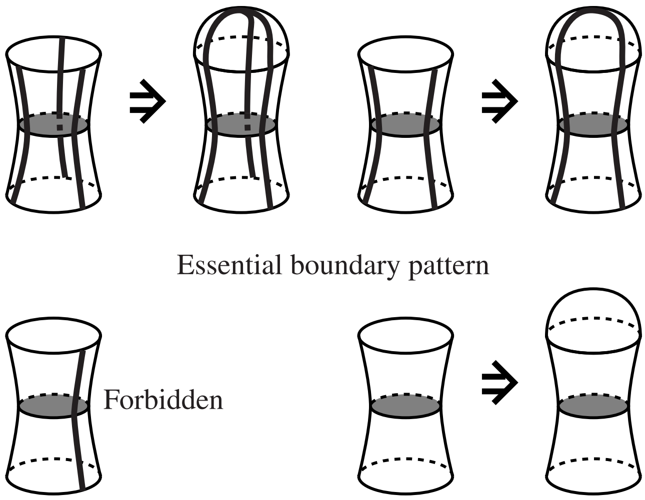


Figure 27.

Definition. A boundary pattern P is *homotopically essential* if, for each map of a disc $(D, \partial D) \to (M, \partial M)$ with $\partial D \cap P$ at most three points (which are disjoint), there is a homotopy (keeping $\partial D \cap P$ fixed, introducing no new points of $\partial D \cap P$, and keeping ∂D in ∂M) to an embedding of D into ∂M so that the image of D contains at most one vertex of P and no simple closed curves of P.

Clearly, if a boundary pattern is homotopically essential, then it is essential. (A proof of this requires the fact from surface topology that if two properly embedded arcs in a surface are homotopic keeping their endpoints fixed, then they are ambient isotopic keeping their endpoints fixed.) The main technical result that we will prove is that the converse holds.

Theorem 9.1. *An essential boundary pattern for a compact orientable irreducible 3-manifold is homotopically essential.*

The Loop Theorem is a corollary of this result. This remarkable result is one of the most important theorems in 3-manifold theory. In this course, we will give a new proof of it, using hierarchies.

Theorem 9.2. (The Loop Theorem) *Let M be a compact orientable irreducible 3-manifold. Then ∂M is incompressible if and only if $\pi_1(F) \rightarrow \pi_1(M)$ is injective for each component F of ∂M.*

Proof of 9.2 from 9.1. A standard fact from surface topology gives that a simple closed curve in ∂M is homotopically trivial in ∂M if and only if it bounds a disc in ∂M. Hence, if a component F of ∂M is compressible, then $\pi_1(F) \rightarrow \pi_1(M)$ is not injective.

To prove the converse, suppose that ∂M is incompressible. Let P be the empty boundary pattern in ∂M. This is then essential. By Theorem 9.1, P is homotopically essential. Hence, if ℓ is any loop in ∂M that is homotopically trivial in M, then ℓ is homotopically trivial in ∂M. □

We can in fact prove the following slightly stronger version of the Loop Theorem.

Theorem 9.3. *Let M be a compact orientable irreducible 3-manifold, and let F be a connected surface in ∂M. If $\pi_1(F) \rightarrow \pi_1(M)$ is not injective, then F is compressible.*

Proof of 9.3 from 9.1. Suppose that F is incompressible. Let ∂F be the boundary pattern of M. If this is not essential, then there is a compressing disc for ∂M that intersects ∂F at most twice. Decompose M along this disc to give a new 3-manifold M'. Let F' be $M' \cap F$. Then $\pi_1(F) \to \pi_1(M)$ is injective if and only if each component of F' is π_1-injective in M'. Also, F' is incompressible in M'. Repeat this process if necessary. At each stage, we reduce the complexity of ∂M. Hence, we may assume that the boundary pattern ∂F is essential in M. By Theorem 9.1, it is homotopically essential, and therefore $\pi_1(F) \to \pi_1(M)$ is injective. □

This stronger version of Theorem 9.3 allows us to prove Theorem 3.3.

Theorem 3.3. *Let S be a connected compact orientable surface properly embedded in a compact orientable irreducible 3-manifold M. Then S is incompressible if and only if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is an injection.*

Proof. Suppose that $\pi_1(S) \to \pi_1(M)$ is not injective. There is then a map $h\colon (D, \partial D) \to (M, S)$ of a disc D such that $h(\partial D)$ is homotopically non-trivial in S. Using an argument similar to that in Lemma 7.8, we may perform a homotopy of D (keeping ∂D fixed) so that $h^{-1}(S)$ is a collection of simple closed curves in D. Pick one innermost in D. If this is sent to a curve that is homotopically trivial in S, we may modify h and remove this curve. Hence, we may assume that there is a map $h\colon D \to M$ so that $h^{-1}(S) = \partial D$ and so that $h(\partial D)$ is homotopically non-trivial in S. We may also assume that $h|_{\mathcal{N}(\partial D)}$ respects the product structure on $\mathcal{N}(S)$. Hence, h restricts to a trivialising homotopy for some loop in one of the two copies of S in M_S. Applying Theorem 9.3 to this copy F of S gives that F is compressible. Extending the compression disc using the product structure $\mathcal{N}(S) \cong S \times I$ gives a compression disc for S. □

Remark. This argument fails (and the result need not be true) when S is non-orientable: since $\mathcal{N}(S)$ is not a product, a compression disc for the ∂I-bundle of $\mathcal{N}(S)$ does not necessarily extend to a compression for S.

Theorem 9.3, together with the existence of hierarchies, also allows us to prove the following.

Theorem 9.4. *Let M be a compact orientable Haken 3-manifold. Then $\pi_k(M) = 0$ for all $k \geq 2$.*

Proof. Pick a hierarchy

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n.$$

Consider a map $h: S^2 \to M$, and let S_i be the first surface to intersect $h(S^2)$. We may homotope h so that $h^{-1}(S_i)$ is a collection of simple closed curves. Let C be one innermost in S^2 bounding a disc D. Then $h(C)$ is homotopically trivial in M_{i+1}. Hence, by the argument of Theorem 9.3, we may homotope D into S_i. There is then a a further homotopy removing C from $h^{-1}(S_i)$. We may therefore assume that $h(S^2) \subset M_{i+1}$. Repeating this as far as M_n gives that $h(S^2) \subset M_n$. Since $\pi_2(M_n)$ is trivial, h represents a trivial element of $\pi_2(M)$. Therefore $\pi_2(M) = 0$.

If M is closed, then $\pi_1(M)$ contains the fundamental group of a closed orientable surface other than a 2-sphere, and hence $\pi_1(M)$ is infinite. If M has non-empty boundary, then (providing it is not a 3-ball), $H_1(M)$ is infinite, by Theorem 7.5, and so $\pi_1(M)$ is infinite. Therefore the universal cover $\tilde{M}$ of M is non-compact. Hence, $H_k(\tilde{M}) = 0$ for all $k \geq 3$. Now, $\pi_k(\tilde{M}) \cong \pi_k(M)$ for all $k \geq 2$. Therefore, $\pi_2(\tilde{M}) = 0$. Hence, by the Hurewicz theorem, $\pi_k(\tilde{M}) \cong H_k(\tilde{M}) \cong 0$ for all $k \geq 3$. This proves the theorem. □

Remark. It is possible to show (using rather different methods) that $\pi_2(M) = 0$ for all irreducible orientable 3-manifolds M. Hence, if in addition $\pi_1(M)$ is infinite, $\pi_k(M) = 0$ for all $k \geq 3$.

10. Special hierarchies

Definition. Let S be a surface properly embedded in a 3-manifold M with boundary pattern P. Then a *pattern-compression disc* for S is a disc D embedded in M such that

- $D \cap S$ is an arc α in ∂D,
- $\partial D - \text{int}(\alpha) = D \cap \partial M$ intersects P at most once, and
- α does not separate off a disc from S intersecting P at most once.

If no such pattern-compression disc exists, then S is *pattern-incompressible*.

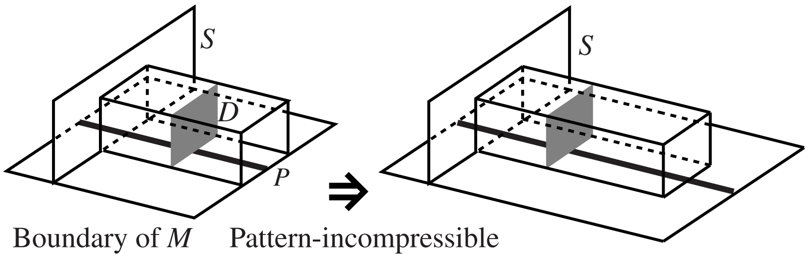


Figure 28.

Definition. Two surfaces S_0 and S_1 embedded in a 3-manifold M are *parallel* if there is an embedding of $S \times [0,1]$ in M such that $S_0 = S \times \{0\}$ and $S_1 = S \times \{1\}$. If $\partial(S \times [0,1]) - S_0 \subset \partial M$, we say that S_0 is *boundary-parallel*.

Definition. A *special hierarchy* for a compact orientable irreducible 3-manifold M with boundary pattern P is a hierarchy for M of properly embedded connected pattern-incompressible incompressible surfaces, none of which is a 2-sphere or boundary-parallel disc. (At each stage, the cut-open 3-manifold inherits its boundary pattern from the previous one.) We write the manifolds and boundary patterns as:

$$(M,P) = (M_1,P_1) \xrightarrow{S_1} (M_2,P_2) \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} (M_n,P_n).$$

We now give an overview of the proof of Theorem 9.1. It proceeds in four main steps:

1. Show that any compact connected orientable irreducible 3-manifold M with essential boundary pattern P and non-empty boundary has a special hierarchy

$$(M,P) = (M_1,P_1) \xrightarrow{S_1} (M_2,P_2) \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} (M_n,P_n).$$

2. Show that (M_i,P_i) is essential if and only if (M_{i+1},P_{i+1}) is.
3. Show, using simple properties of the 3-ball, that (M_n,P_n) being essential implies that it is homotopically essential.
4. Show that if (M_{i+1},P_{i+1}) is homotopically essential, then so is (M_i,P_i).

We will save step 1 until §11. We now embark on steps 2, 3 and 4.

Lemma 10.1. *Let M be a compact orientable irreducible 3-manifold with essential boundary pattern P. Let S be a connected pattern-incompressible incom-*

pressible surface in M, which is not a boundary-parallel disc. Then the 3-manifold M_S obtained by cutting along S inherits an essential boundary pattern P'.

Proof. Let D be a disc properly embedded in M_S with $\partial D \cap P'$ at most three points. The curve ∂D may run through parts of ∂M_S coming from ∂M and parts coming from S. Note however the points where it swaps must be points of $\partial D \cap P'$, and that at most one side of any point of $\partial D \cap P'$ lies in S. Hence, at most one arc of $\partial D - P'$ lies in S.

Case 1. ∂D is disjoint from S.

Then $\partial D \subset \partial M$. Since P is essential, ∂D bounds a disc D' in ∂M containing at most one vertex of P and no simple closed curves. If S intersects D', then pick a simple closed curve of $S \cap D'$ innermost in D'. The disc this bounds cannot be a compression disc for S. Hence, S must be a disc. Since M is irreducible, it is parallel to a disc in ∂M, contrary to assumption. Hence, D' is disjoint from S, and therefore lies in ∂M_S. This verifies that D does not violate the essentiality of P'.

Case 2. ∂D intersects S.

Then $\partial D - S$ intersects P at most once. Since D is not a pattern-compressing disc for S, $D \cap S$ separates off a disc D_1 of S intersecting P in at most one point. Then, $D \cup D_1$ is a disc properly embedded in M, intersecting P in at most two points. There is therefore a disc D_2 in ∂M with $\partial D_2 = \partial(D \cup D_1)$, containing at most one vertex of P and no simple closed curves, since P is essential. Since $D \cup D_1$ intersects P in at most two points, D_2 cannot therefore contain any vertex of P. Therefore, $D_1 \cup D_2$ is a disc in ∂M_S containing at most one vertex of P' and no simple closed curves. This gives that P' is essential. □

Lemma 10.2. *Suppose that M is a 3-ball with essential boundary pattern P. Then P is homotopically essential.*

Proof. Consider a map $(D, \partial D) \to (M, \partial M)$ with $\partial D \cap P$ at most three points. Since P is essential, each component of $\partial M - P$ is a disc. We may therefore homotope each arc of $\partial D - P$ so that it is embedded. The arcs $\partial D - P$ lie in different components of $\partial M - P$, since P is a boundary pattern. Hence, we have homotoped ∂D so that it is embedded. It therefore bounds an embedded disc D'

in ∂M. Since P is essential, D' contains at most one vertex of P and no simple closed curves. As the 3-ball has trivial π_2, there is a homotopy taking D to D' keeping ∂D fixed. □

Lemma 10.3. *Let M be a compact orientable 3-manifold with boundary pattern P. Let S be an orientable incompressible pattern-incompressible surface properly embedded in M. Let P' be the boundary pattern inherited by M_S. If P' is homotopically essential, then so is P.*

Proof. Consider a map $h\colon (D, \partial D) \to (M, \partial M)$ such that ∂D intersects P in at most three points. We may perform a small homotopy so that $h^{-1}(S)$ is a collection of properly embedded arcs and circles in D.

Suppose that there is some simple closed curve of $h^{-1}(S)$. Pick one C innermost in D, bounding a disc D'. Since P' is homotopically essential, we may homotope D' to an embedded disc in S. Perform a further small homotopy to reduce $|h^{-1}(S)|$.

Hence, we may assume that there are no simple closed curves of $h^{-1}(S)$. If there is more than one arc, at least two are extrememost in D. They separate off discs D_1 and D_2 from D. Similarly, if there is only one arc of $h^{-1}(S)$, it divides D into two discs D_1 and D_2. There are only three points of $h^{-1}(P)$, and so D_1, say, contains at most one of these points. Hence, $h(\partial D_1)$ intersects P' in at most three points. Since P' is homotopically essential, we may homotope D_1 to an embedded disc D' in ∂M_S containing at most one vertex of P' and no simple closed curves. Replace D_1 with D', and perform a homotopy to reduce $|h^{-1}(S)|$.

Repeat this process until $h^{-1}(S) = \emptyset$. Then, use that P' is homotopically essential to construct the desired homotopy of D to an embedded disc in ∂M containing at most one vertex of P and no simple closed curves. □

This completes steps 2, 3 and 4. A similar argument to that of Lemma 10.3 gives the following.

Lemma 10.4. *Let M be a compact orientable 3-manifold with boundary pattern P. Let S be an orientable incompressible pattern-incompressible surface properly embedded in M. Let P' be the boundary pattern inherited by M_S. If P' is essential, then so is P.*

All that is now required in the proof of the Loop Theorem is to establish the existence of special hierarchies. For this, we need extra machinery.

11. Normal surfaces

Definition. A *triangle* (respectively, *square*) in a 3-simplex Δ^3 is a properly embedded disc D such that ∂D intersects precisely three (respectively, four) 1-simplices transversely in a single point, and is disjoint from the remaining 1-simplices and all the vertices.

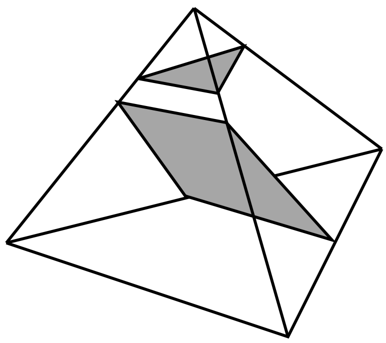


Figure 29.

Fix a triangulation T of the 3-manifold M.

Definition. A properly embedded surface in M is *in normal form* with respect to T if it intersects each 3-simplex in a finite (possibly empty) collection of disjoint triangles and squares.

Theorem 11.1. *Let M be a compact irreducible 3-manifold. Let S be a properly embedded closed incompressible surface in M, with no component of S a 2-sphere. Then, for any triangulation T of M, S may ambient isotoped into normal form.*

Proof. First, a small ambient isotopy makes S transverse to the 2-skeleton of the triangulation. Then S intersects each 2-simplex in a collection of arcs and simple closed curves. We may assume that it misses the vertices of T. Let the *weight* $w(S)$ of S be the number of intersections between S and the 1-simplices.

Suppose first that there is a simple closed curve of intersection between S and the interior of some 2-simplex. Pick one C innermost in the 2-simplex, bounding a disc D in the 2-simplex. Then C bounds a disc D' in S, since D is not a compression disc for S. Since M is irreducible, we may ambient isotope D' onto D. This does not increase $w(S)$. Hence, we may assume that S intersects each 2-simplex in a (possibly empty) collection of arcs.

If $w(S)$ is zero, then each component of S lies in a 3-simplex. By Theorem 3.8, any such component is 2-sphere, contrary to assumption. We will perform a sequence of ambient isotopies to the surface, which will reduce $w(S)$ and hence are guaranteed to terminate.

Let Δ^3 be a 3-simplex of M. Suppose first that S intersects Δ^3 in something other than a collection of discs. If there is a non-disc component of $S \cap \Delta^3$ with non-empty boundary, then pick a curve of $S \cap \partial\Delta^3$ innermost in $\partial\Delta^3$ among all curves not bounding discs of $S \cap \Delta^3$. This bounds a compression disc D for $S \cap \Delta^3$. Since S is incompressible in M, ∂D bounds a disc in S. Ambient isotope this disc onto D to decrease $w(S)$. If every component of $S \cap \Delta^3$ with non-empty boundary is disc, then any closed component of $S \cap \Delta^3$ lies in the complement of these discs, which is a 3-ball. Hence, it is a 2-sphere by Theorem 3.8. Thus, we may assume that each component of $S \cap \Delta^3$ is a disc.

Now suppose that some disc D of $S \cap \Delta^3$ intersects a 1-simplex σ more than once, as in Figure 30. We claim that we can find such a disc D, and two points of $D \cap \sigma$, so that no other points of $S \cap \sigma$ lie between them on σ. First pick two points of $D \cap \sigma$ having no points of $D \cap \sigma$ between them on σ. Let β be the arc of σ between them. Note that ∂D separates $\partial\Delta^3$ into two discs and that β is properly embedded in one of these. Hence, if D' is any other disc of $S \cap \Delta^3$, it intersects β in an even number of points. Hence, we may find a disc D of $S \cap \Delta^3$ intersecting σ in adjacent points on σ. Let β be the arc of σ between them, and let α be some arc properly embedded in D joining these two points. Note that $S \cap \Delta^3$ separates Δ^3 into 3-balls and that $\alpha \cup \beta$ lies in the boundary of one of these balls. Hence, there is a disc D' embedded in Δ^3 with $D' \cap (S \cup \partial\Delta^3) = \alpha \cup \beta$. Then we may use the disc D' to ambient isotope S, reducing $w(S)$, as in Figure 30.

Hence, we may assume that each disc of $S \cap \Delta^3$ intersects each 1-simplex at most once. It is then a triangle or square. Hence, S is now normal. □

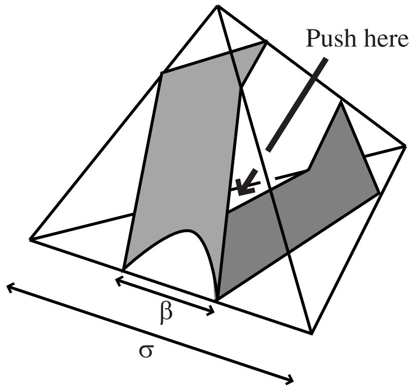


Figure 30.

Theorem 11.2. *Let M be a compact orientable irreducible 3-manifold. Then there is some integer $n(M)$ with the following property. If S is a closed properly embedded incompressible surface in M with more than $n(M)$ components, none of which is a 2-sphere, then at least two components of S are parallel (with no component of S in the product region between them).*

Proof. We let $n(M) = 2\beta_1(M; \mathbb{Z}_2) + 6t$, where t is the number 3-simplices in some triangulation of M. Let S have components $S_1, \dots, S_k$, with $k > n(M)$. Then, by Theorem 11.1, S may be ambient isotoped into normal form. Note that M_S has more than $\beta_1(M; \mathbb{Z}_2) + 6t$ components. Also, for each 3-simplex Δ^3, all but at most six components of $\Delta^3 - S$ is a product region, lying between adjacent triangles or squares. Therefore, more than $\beta_1(M; \mathbb{Z}_2)$ components of M_S are composed entirely of product regions. Each such component X of M_S is an I-bundle. If X is not a product I-bundle, then it is an I-bundle over a non-orientable surface. Then we can calculate that $H_1(\partial X; \mathbb{Z}_2) \to H_1(X; \mathbb{Z}_2)$ is not surjective. Hence, there is a non-trivial summand of $H_1(M; \mathbb{Z}_2)$ for each such component X of M. So, at most $\beta_1(M; \mathbb{Z}_2)$ are of this form. Hence, there is at least one product I-bundle of M_S. Its two boundary components are parallel in M. □

Lemma 11.3. *Let M be a compact orientable 3-manifold, and let*

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n$$

be a partial hierarchy. Let $X = \mathcal{N}(\partial M \cup S_1 \cup \dots \cup S_{n-1})$. Then $\partial X - \partial M$ is incompressible in X.

Proof. Consider a compression disc D for $\partial X - \partial M$ in X. Let S_i be the first surface in the hierarchy it intersects. Then we may assume $D \cap S_i$ is a collection of simple closed curves in the interior of D. Pick one innermost in D, bounding a disc D_1. This cannot be a compression disc for S_i, and so it bounds a disc D_2 in S_i. Remove D_1 from D, replace it with D_2, and perform a small isotopy to reduce $|D \cap S_i|$. This does not introduce any new intersections with $S_1 \cup \ldots \cup S_{i-1}$. Thus, we may assume that D is disjoint from S_i, and, repeating, from all of the surfaces in the partial hierarchy. It therefore lies in the the space X, with the interior of a small regular neighbourhood of $S_1 \cup \ldots \cup S_{n-1}$ removed. This is a copy of $F \times I$, for a closed orientable surface F, with $F \times \{1\}$ identified with $\partial X - \partial M$. But the boundary of $F \times I$ is π_1-injective, and hence incompressible, which is a contradiction. □

Theorem 11.4. *Let M be a compact orientable irreducible 3-manifold with non-empty boundary and an essential boundary pattern P. Then M has a special hierarchy. Furthermore, if M has non-empty boundary, we may assume that no surface in this hierarchy is closed.*

Proof. Suppose first that ∂M is compressible. Let D be a compression disc. If there is a pattern-compression disc for D, then 'compressing' D along this disc decomposes D into two discs. Both of these discs have fewer intersections with P, and at least one of these is a compression disc for ∂M. Focus on this disc, and repeat until we have a pattern-incompressible compression disc for M. Decompose M along this disc. By Lemma 10.1, the resulting manifold M_2 inherits an essential boundary pattern. If its boundary is compressible, cut again along a pattern-incompressible compression disc. Repeat, giving a partial special hierarchy

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \ldots \xrightarrow{S_{i-1}} M_i,$$

where ∂M_i is incompressible in M_i. We must reach such an M_i, since the complexity of ∂M_2 is less than that of ∂M_1, and so on. Push ∂M_i a little into M, giving a closed properly embedded surface F_1.

Claim. F_1 is incompressible in M.

The surface F_1 separates M into two components: M_i and $X = \mathcal{N}(\partial M \cup S_1 \cup \ldots \cup S_{i-1})$. By assumption, F_1 is incompressible in M_i. By Lemma 11.3, F_1 is incompressible in X. This proves the claim.

Each 2-sphere component of ∂M_i bounds a 3-ball. If every component of ∂M_i is a 3-ball, then we have constructed our special hierarchy as required. Suppose therefore that at least one component of ∂M_i is not a 2-sphere. By Theorem 7.6, M_i contains a properly embedded connected incompressible 2-sided non-separating surface S. If M_i has non-empty boundary, then we may assume that ∂S is non-empty. If S has a pattern-compression disc, then 'compress' S along this disc giving a surface S'. Then S' is incompressible and 2-sided, and at least one component S_1 of S' is non-separating. Then either $\chi(S_1) > \chi(S)$, or $\chi(S_1) = \chi(S)$ and $|S_1 \cap P| < |S \cap P|$. Hence, we may assume that S is pattern-incompressible. Cut along this surface to give M_{i+1}. If ∂M_{i+1} is compressible, then, as above, compress it as far as possible to give a closed incompressible surface F_2 in M. Note that F_1 and F_2 are disjoint. Continue this process. If we have not stopped by the time we have constructed $F_{n(M)+1}$, Theorem 11.2 implies that at some stage F_i and F_j are parallel for some $i < j$, with no F_k in the product region between them. Some S_p lies in this product region. The theorem is then proved by the following lemma. □

Lemma 11.5. *Let F be a compact orientable surface. Then there is no connected non-separating incompressible surface S properly embedded in $F \times [0,1]$ that is disjoint from $F \times \{1\}$.*

Proof. If F is closed, pick a simple closed curve C in F that does not bound a disc. Then $C \times [0,1]$ is an annulus A. A small ambient isotopy of S ensures that $S \cap A$ is a collection of arcs and simple closed curves. We may remove all simple closed curves of $S \cap A$ that bound discs in A. If there is an arc, it has both its endpoints in $C \times \{0\}$. We may find such an arc separating off a disc of A with interior disjoint from S. 'Compress' S along this disc to reduce $|S \cap A|$. The result is still an incompressible surface, and at least one component is non-separating. Hence, we may assume that $S \cap A$ contains only simple closed curves. By 'compressing' S along annuli in A, we may remove each of these. Hence, we may assume that S lies in $(F - C) \times [0,1]$. Therefore, we may assume that F has non-empty boundary. Pick a collection α of arcs properly embedded in F which cut F to a disc. Apply an argument as above to ensure that S is disjoint from $\alpha \times [0,1]$. It is then a disc properly embedded in $(F - \mathcal{N}(\alpha)) \times [0,1]$, which is a 3-ball. It is therefore separating. □

12. Topological rigidity

In this section, we will prove that homotopy equivalent closed Haken 3-manifolds are homeomorphic. The main ingredients are the existence of hierarchies and the loop theorem. A vital part of the argument is a version of topological rigidity for surfaces. Its proof is instructive, since it follows the same approach as the 3-manifold case.

Theorem 12.1. *Let F and G be connected compact surfaces with $\pi_1(F) \neq 0$. Let $f\colon (F, \partial F) \to (G, \partial G)$ be a map with $f_*\colon \pi_1(F) \to \pi_1(G)$ injective. Then, there is a homotopy through maps $f_t\colon (F, \partial F) \to (G, \partial G)$ with $f_0 = f$ and either*

(i) $f_1\colon F \to G$ is a covering map, or

(ii) F is an annulus or Möbius band and $f_1(F) \subset \partial G$.

If, for some components C of ∂F, $f|_C$ is a covering map, we can require that $f_t|_C = f|_C$ for all t.

Lemma 12.2. *Let $f\colon (F, \partial F) \to (G, \partial G)$ be a map between connected surfaces with non-empty boundary such that*

1. *$f|_{\partial F}$ is not injective, and its restriction to each component of ∂F is a cover,*
2. *$f_*\colon \pi_1(F) \to \pi_1(G)$ is an isomorphism,*
3. *$\pi_1(F) \neq 0$, and*
4. *F is compact.*

Then conclusion (ii) of Theorem 12.1 holds.

Proof. By (1), there are two points in ∂F mapping to the same point in ∂G, and there is a path $\gamma\colon I \to F$ joining them. Then $f \circ \gamma$ is a loop in G. By (2), there is a loop β in F based at $\gamma(0)$ such that $f_*([\beta]) = [f \circ \gamma]^{-1} \in \pi_1(G, f\gamma(0))$. Then $\alpha = \beta.\gamma$ is a path $(I, \partial I) \to (F, \partial F)$ such that $\alpha(0) \neq \alpha(1)$ and $f \circ \alpha$ is a homotopically trivial loop in G.

For $i = 0$ and 1, let J_i be the component of ∂F containing $\alpha(i)$. (Possibly, $J_0 = J_1$.) Orient J_i in some way, so that it is a loop based at $\alpha(i)$. Let K be the component of ∂G containing $f \circ \alpha(0) = f \circ \alpha(1)$. Then $f_*([J_0])$ and $f_*([\alpha.J_1.\alpha^{-1}])$

are both non-zero powers of $[K]$ in $\pi_1(G, f\alpha(0))$, by (1). Hence, by (2), some power of $[J_0]$ is some power of $[\alpha.J_1.\alpha^{-1}]$ in $\pi_1(F, \alpha(0))$. Let $x = \alpha(0)$. Let $p\colon (\tilde{F}, \tilde{x}) \to (F, x)$ be the covering of F such that $p_*\pi_1(\tilde{F}, \tilde{x}) = i_*\pi_1(J_0, x)$, where $i\colon J_0 \to F$ is the inclusion map. Lift α to a path $\tilde{\alpha}$ starting at $\tilde{x}$. Let $\tilde{J}_i$ be the component of $\partial\tilde{F}$ containing $\tilde{\alpha}(i)$. Since some power of $[\alpha.J_1.\alpha^{-1}]$ is some power of $[J_0] \in \pi_1(F, x)$, $\tilde{J}_1$ is compact.

Claim. $\tilde{J}_0 \neq \tilde{J}_1$.

Otherwise, since $\pi_1(\tilde{J}_0) \to \pi_1(\tilde{F})$ is an isomorphism, we may homotope $\tilde{\alpha}$ (keeping its endpoints fixed) to a path α_1 in $\tilde{J}_0$. But then $f \circ p \circ \alpha_1$ is a loop in K which lifts to a path under the covering $f|_{J_0}\colon J_0 \to f(J_0) \subset K$. Since $f \circ p \circ \alpha_1$ is null-homotopic in G, $\pi_1(K) \to \pi_1(G)$ is therefore not injective. Hence G is a disc and so, by (2), $\pi_1(F) = 0$. However, this contradicts (3) and so this proves the claim.

Claim. $\tilde{F}$ is compact.

We have the following exact sequence:

$$0 \to H_2(\tilde{F}, \tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2) \to H_1(\tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2) \to H_1(\tilde{F}; \mathbb{Z}_2).$$

The last of the above groups is isomorphic to $H_1(\tilde{J}_0; \mathbb{Z}_2) \cong \mathbb{Z}_2$. The middle group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence, the first group must be non-trivial. Hence, $\tilde{F}$ is a compact surface.

The only compact surface with the property that some power of one boundary component can be freely homotoped into one power of another boundary component is an annulus. Since $\chi(\tilde{F})$ is a multiple of $\chi(F)$, F is an annulus or Möbius band. Using that $f \circ \alpha$ is homotopically trivial, we can retract f into ∂G. So (ii) of Theorem 12.1 holds. □

Proof of Theorem 12.1. Let $p\colon \tilde{G} \to G$ be the cover where $p_*\pi_1(\tilde{G}) = f_*\pi_1(F)$. Construct a lift

$$\begin{array}{ccc} & & \tilde{G} \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ F & \xrightarrow{f} & G \end{array}$$

Then $\tilde{f}_*$ is an isomorphism. We will show that $\tilde{f}$ may homotoped so that either (i) or (ii) hold. This will prove the result.

Note that each boundary component of F is π_1-injective in F. Hence, if $\tilde{f}$ is not already a covering map on ∂F, we may homotope it to so that it is a cover. If $\tilde{f}|_{\partial F}$ is not a homeomorphism, then by Lemma 12.1, case (ii) of Theorem 12.1 holds for $\tilde{f}$ and hence f. So, we may assume that $\tilde{f}|_{\partial F}$ is a homeomorphism onto its image.

Claim. $\tilde{G}$ is compact.

If $\tilde{G}$ is non-compact, then $\pi_1(\tilde{G})$ is free. So, F is not a closed surface. Note that the following commutes

$$\begin{array}{ccc} H_2(F, \partial F; \mathbb{Z}_2) & \longrightarrow & H_1(\partial F; \mathbb{Z}_2) \\ \Big\downarrow & & \Big\downarrow \\ H_2(\tilde{G}, \partial\tilde{G}; \mathbb{Z}_2) & \longrightarrow & H_1(\partial\tilde{G}; \mathbb{Z}_2) \end{array}$$

Since the map along the top has non-zero image and the map on the right is injective, their composition is not the zero map. Hence, $H_2(\tilde{G}, \partial\tilde{G}; \mathbb{Z}_2)$ is non-trivial and so $\tilde{G}$ is compact.

By looking at $\tilde{f}$ instead of f, it therefore suffices to consider the case where f_* is an isomorphism and $f|_{\partial F}$ is a homeomorphism onto its image. Consider first the case where ∂G is non-empty. Pick a collection A of properly embedded arcs in G which cut it to a disc. We may homotope f (keeping it unchanged on ∂F) so that $f^{-1}(A)$ is a collection of properly embedded arcs and simple closed curves. If there is any simple closed curve, its image in G lies in an arc, and hence is homotopically trivial. Hence, each simple closed curve of $f^{-1}(A)$ bounds a disc. By repeatedly considering an innermost such curve, we may homotope f to remove all such simple closed curves.

Since $f|_{\partial F}$ is a homeomorphism, the endpoints of each arc of $f^{-1}(A)$ map to distinct points in G. Hence, we may homotope $f|_{\mathcal{N}(f^{-1}(A)\cup\partial F)}$ so that it is a homeomorphism. But the remainder $F-(f^{-1}(A)\cup\partial F)$ maps to a disc in G. Since f is π_1-injective, $F-(f^{-1}(A)\cup F)$ is a collection of discs. A map of a disc to a disc that is a homeomorphism from boundary to boundary may be homotoped to a homeomorphism. Hence, we have therefore homotoped f to a homeomorphism.

Now consider the case where G is closed. Pick a simple closed curve C in G that does not bound a disc. Homotope f so that $f^{-1}(C)$ is a collection of simple closed curves in F, none of which bounds discs. Then $f|_{F-\mathrm{int}(\mathcal{N}(f^{-1}(C)))}\colon F-$

$\text{int}(\mathcal{N}(f^{-1}(C))) \to G - \text{int}(\mathcal{N}(C))$ is π_1-injective. We have proved the theorem in the case of surfaces with non-empty boundary. Consider therefore a component of $F - \text{int}(\mathcal{N}(f^{-1}(C)))$. If it is an annulus or Möbius band that can be homotoped into C, then perform this homotopy. A further small homotopy reduces the number of components of $f^{-1}(C)$. Hence, we may assume that case (i) applies to each component of $F - \text{int}(\mathcal{N}(f^{-1}(C)))$. Then we have homotoped f to a cover. □

We can now tackle topological rigidity for Haken 3-manifolds. The full result is the following.

Theorem 12.3. *Let M and N be Haken 3-manifolds. Suppose that there is a map $f\colon (M, \partial M) \to (N, \partial N)$ such that $f_*\colon \pi_1(M) \to \pi_1(N)$ is injective, and such that for each component B of ∂M, $(f|_B)_*\colon \pi_1(B) \to \pi_1(B')$ is injective, where B' is the component of ∂N containing $f(B)$. Then there is a homotopy $f_t\colon (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either*

(i) $f_1\colon M \to N$ is a covering map,

(ii) M is an I-bundle over a closed surface and $f_1(M) \subset \partial N$, or

(iii) N and M are solid tori $D^2 \times S^1$ and

$$\begin{aligned} f_1\colon D^2 \times S^1 &\to D^2 \times S^1 \\ (r, \theta, \phi) &\mapsto (r, p\theta + q\phi, s\phi), \end{aligned}$$

where $p, s \in \mathbb{Z} - \{0\}$ and $q \in \mathbb{Z}$.

If, for any components B of ∂M, $f|_B$ is already a cover, then we may assume that $f_t|_B = f|_B$ for all t.

Corollary 12.4. *Let M and N be closed Haken 3-manifolds. Then a homotopy equivalence between them can be homotoped to a homeomorphism.*

In order to prove Theorem 12.3, we will need the following result. Its proof can be found in Chapter 10 of Hempel's book (Theorem 10.6).

Theorem 12.5. *Let M be a compact orientable irreducible 3-manifold, and suppose that $\pi_1(M)$ contains a finite index subgroup isomorphic to the fundamental group of a closed surface other than S^2 or $\mathbb{R}P^2$. Then M is an I-bundle over some closed surface.*

Lemma 12.6. *Suppose that $f\colon (M, \partial M) \to (N, \partial N)$ is a map between connected orientable irreducible 3-manifolds with non-empty boundary such that*

1. *$f|_{\partial M}$ is not injective, and its restriction to each component of ∂M is a cover,*
2. *$f_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism, and*
3. *M is compact.*

Then either (ii) or (iii) of Theorem 12.3 holds.

Proof. This proof was omitted in the lectures. The argument is very similar to that of Lemma 12.2. By (1), there are two points in ∂M mapping to the same point in ∂N, and there is a path $\gamma\colon I \to M$ joining them. Then $f \circ \gamma$ is a loop in N. By (2), there is a loop β in M based at $\gamma(0)$ such that $f_*([\beta]) = [f \circ \gamma]^{-1} \in \pi_1(N, f\gamma(0))$. Then $\alpha = \beta.\gamma$ is a path $(I, \partial I) \to (M, \partial M)$ such that

(∗) $\alpha(0) \neq \alpha(1)$ and $f \circ \alpha$ is a homotopically trivial loop in N.

For $i = 0$ and 1, let J_i be the component of ∂M containing $\alpha(i) = x_i$. (Possibly, $J_0 = J_1$.) Let K be the component of ∂N containing $y = f \circ \alpha(0) = f \circ \alpha(1)$. Let $p\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ be the covering of M such that $p_*\pi_1(\tilde{M}, \tilde{x}_0) = i_{0*}\pi_1(J_0, x_0)$, where $i_0\colon J_0 \to M$ is the inclusion map. Lift α to a path $\tilde{\alpha}$ starting at $\tilde{x}_0$ and ending at $\tilde{x}_1$, say. Let $\tilde{J}_i$ be the component of $\partial\tilde{M}$ containing $\tilde{\alpha}(i)$. There is a commutative diagram

$$\begin{array}{ccc}
\pi_1(J_0, x_0) & \xrightarrow{(f|_{J_0})_*} & \pi_1(K, y) \\
\big\downarrow{\scriptstyle i_{0*}} & & \big\downarrow \\
\pi_1(M, x_0) & \xrightarrow{f_*} & \pi_1(N, y) \\
\big\uparrow{\scriptstyle \psi_\alpha \circ i_{1*}} & & \big\uparrow \\
\pi_1(J_1, x_1) & \xrightarrow{(f|_{J_1})_*} & \pi_1(K, y)
\end{array}$$

where i_0 and i_1 are the relevant inclusion maps, and ψ_α is the 'change of basepoint map' $\pi_1(M, x_1) \to \pi_1(M, x_0)$ sending a loop ℓ based at x_1 to $\alpha.\ell.\alpha^{-1}$. Commutativity of the lower half of the diagram follows from the fact that $f \circ \alpha$ is homotopically trivial. Since $f|_{J_i}$ is a finite sheeted covering, we conclude that $\psi_\alpha i_{1*}\pi_1(J_1, x_1) \cap i_{0*}\pi_1(J_0, x_0)$ has finite index in each term. This intersection is $p_*\psi_\alpha \tilde{i}_{1*}\pi_1(\tilde{J}_1, \tilde{x}_1)$, where $\tilde{i}_1\colon J_1 \to \tilde{M}$ is the inclusion map. Hence, we conclude that $\tilde{J}_1$ is compact and that a nonzero power of each loop in $\tilde{J}_0$ is freely homotopic in $\tilde{M}$ to a loop in $\tilde{J}_1$. Note also that $p|_{\tilde{J}_0}\colon \tilde{J}_0 \to J_0$ is a homeomorphism.

Case 1. There is some path α satisfying $(*)$ which also satisfies

$(**)$ α is not homotopic (keeping $\partial\alpha$ fixed) to a path in ∂M.

Then $\tilde{J}_0 \neq \tilde{J}_1$. Otherwise, since $\pi_1(\tilde{J}_0) \to \pi_1(\tilde{M})$ is surjective, $\tilde{\alpha}$ would homotope into $\tilde{J}_0$ and projecting this homotopy would contradict $(**)$. In addition, we can conclude that $\tilde{J}_0$ is incompressible in $\tilde{M}$. If not, we could write $\pi_1(\tilde{M})$ as a free product, with $\pi_1(\tilde{J}_1)$ conjugate to a subgroup of one factor. This is not possible, since $\pi_1(\tilde{J}_1)$ maps to a subgroup of finite index in $\pi_1(\tilde{M})$. Thus, $\tilde{i}_{0*}\colon \pi_1(\tilde{J}_0) \to \pi_1(\tilde{M})$ is injective, and therefore an isomorphism. Hence, $\tilde{i}_0$ is a homotopy equivalence, as all the higher homotopy groups of $\tilde{J}_0$ and $\tilde{M}$ are trivial. We have the exact sequence

$$0 \to H_3(\tilde{M}, \tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2) \to H_2(\tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2) \to H_2(\tilde{M}; \mathbb{Z}_2).$$

Since $H_2(\tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_2(\tilde{M}; \mathbb{Z}_2) \cong H_2(\tilde{J}_0; \mathbb{Z}_2) \cong \mathbb{Z}_2$, we deduce that $H_3(\tilde{M}, \tilde{J}_0 \cup \tilde{J}_1; \mathbb{Z}_2)$ is non-trivial, and hence $\tilde{M}$ is compact. Hence, $i_{0*}\pi_1(J_0)$ has finite index in $\pi_1(M)$. By Theorem 12.5, M is an I-bundle over a closed surface.

We now obtain a homotopy retracting M into ∂N. The map $i_{0*}\colon \pi_1(J_0) \to \pi_1(M)$ is a injection. For otherwise, J_0 is compressible and hence so is $\tilde{J}_0$, which we already know not be the case. This implies that $\pi_1(K) \to \pi_1(N)$ is an injection. For if some element of $\pi_1(K)$ were sent to the identity in $\pi_1(N)$, then some power of it would lie in the image of $\pi_1(J_0)$ and hence J_0 would not be π_1-injective. Consider the covering $q\colon \tilde{N} \to N$ corresponding to $f_*\pi_1(J_0)$. An appropriate lifting $\tilde{f}$ of f takes J_0 and J_1 into a component $\tilde{K}$ of $q^{-1}(K)$ (the same component since $[f \circ \alpha] = 1$). The map $\pi_1(\tilde{K}) \to \pi_1(\tilde{N})$ is necessarily surjective, and it is injective since $\pi_1(K) \to \pi_1(N)$ is injective. All higher homotopy groups of $\tilde{K}$ and $\tilde{N}$ are trivial, and so the inclusion of $\tilde{K}$ into $\tilde{N}$ is a homotopy equivalence. Hence, there is a deformation retract of $\tilde{N}$ onto $\tilde{K}$, by a homotopy $\rho_t\colon \tilde{N} \to \tilde{N}$. Then $f_t = q \circ \rho_t \circ \tilde{f}$ homotopes M into ∂N. Hence we have conclusion (ii) of Theorem 12.3.

Note that if F sends two different components of ∂M to the same component of ∂N, then we may find a path α satisfying $(*)$ and $(**)$. Hence, the theorem holds in this case. On the other hand, if F sends distinct components of ∂M to distinct components of ∂N, then the right-hand map in the following diagram is injective:

$$\begin{array}{ccc} H_3(M, \partial M) & \longrightarrow & H_2(\partial M) \\ \big\downarrow \tilde{f}_* & & \big\downarrow \tilde{f}_* \\ H_3(N, \partial N) & \longrightarrow & H_2(\partial N) \end{array}$$

Thus, $H_3(N, \partial N)$ is non-trivial, and therefore N is compact.

Case 2. No path α satisfies both $(*)$ and $(**)$.

Then every path α satisfying $(*)$ is homotopic (keeping its endpoints fixed) to a path α_1 in ∂M. Hence, $J_0 = J_1$. The loop $f \circ \alpha_1$ is not contractible in K, since $f|_{J_0}$ is a cover onto K. However $f \circ \alpha_1$ is homotopically trivial in N. Therefore, K is compressible in N. We wish to show that K is a torus and deduce (iii) of Theorem 12.3.

If f maps two distinct components of ∂M to the same component of ∂N then there is a path β joining these components such that $f \circ \beta$ is a loop. Since f_* is surjective, we may assume that $[f \circ \beta] = 1$, and hence β satisfies $(*)$ and $(**)$. Therefore, f takes distinct components of ∂M to distinct components of ∂N. Note that $f|_{J_0}$ is not injective, since α satisfies $(*)$.

Now f is a homotopy equivalence, and so

$$\frac{\chi(\partial M)}{2} = \chi(M) = \chi(N) = \frac{\chi(\partial N)}{2}.$$

(Here, we are using the assumption that N is compact.) Let ∂M have components $J_1, \dots, J_k$, and suppose that $f|_{J_i}$ is n_i-sheeted. Then

$$\sum n_i \chi(f(J_i)) = \sum \chi(J_i) = \chi(\partial M) = \chi(\partial N) = \sum \chi(f(J_i)).$$

So, $n_i = 1$ unless $\chi(f(J_i)) = 0$. Since $n_1 > 1$, $\chi(K) = 0$ and so K is a torus. We have already established that K is compressible. Thus N is a solid torus, since this is the only irreducible 3-manifold with a compressible torus boundary component. Also, J_0 is a torus and $\pi_1(J_0) \to \pi_1(M) \cong \pi_1(N) \cong \mathbb{Z}$. Therefore, J_0 is compressible and M is a solid torus. It is now straightforward to homotope f so that is in the form required by (iii) of Theorem 12.3. □

Proof of Theorem 12.3. Consider first the case where ∂N is non-empty. Let

$p\colon \tilde{N} \to N$ be the cover such that $p_*\pi_1(\tilde{N}) = f_*\pi_1(M)$. Consider the lift

$$\begin{array}{ccc} & & \tilde{N} \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ M & \xrightarrow{f} & N \end{array}$$

Then $\tilde{f}_*$ is an isomorphism. We will show that $\tilde{f}$ may homotoped so that either (i), (ii) or (iii) holds. This suffices to prove the theorem. For if (i) holds for $\tilde{f}$, then $p \circ \tilde{f}$ is a covering map. If (ii) holds for $\tilde{f}$, then composing the homotopy with p, we may homotope M into ∂N. Suppose that (iii) holds for $\tilde{f}$. In particular, $\tilde{N}$ is a solid torus. Then, N must have compressible boundary. Since it is irreducible, and has boundary a torus, it must be a solid torus. Therefore, p is a standard finite covering of the solid torus over itself. The composition of this with $\tilde{f}$ is a map as in (iii), as required.

We are assuming that the restriction of f to each boundary component of M is π_1-injective onto its image component of ∂M. Hence, by Theorem 12.1, we may homotope $f|_{\partial M}$ to a covering. So, $\tilde{f}|_{\partial M}$ is a cover. If $\tilde{f}|_{\partial M}$ sends two distinct components of ∂M to the same component of $\partial \tilde{N}$, then, by Lemma 12.6, (ii) or (iii) of 12.3 hold. So, we may assume that $\tilde{f}|_{\partial M}$ sends distinct components of ∂M to distinct components of $\partial \tilde{N}$. Hence, the right-hand map in the following diagram is injective.

$$\begin{array}{ccc} H_3(M, \partial M) & \longrightarrow & H_2(\partial M) \\ \big\downarrow \tilde{f}_* & & \big\downarrow \tilde{f}_* \\ H_3(\tilde{N}, \partial \tilde{N}) & \longrightarrow & H_2(\partial \tilde{N}) \end{array}$$

So, the fundamental class in $H_3(M, \partial M)$ has non-trivial image in $H_3(\tilde{N}, \partial \tilde{N})$ and hence $\tilde{N}$ is compact.

Hence, it suffices to consider the case where f_* is an isomorphism. By Lemma 12.6, we may assume that $f|_{\partial M}$ is a homeomorphism onto ∂N, for otherwise either (ii) or (iii) holds.

Let

$$N = N_1 \xrightarrow{S_1} N_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} \dots N_n$$

be a hierarchy. By Theorem 11.4, we may assume that each surface has non-empty boundary. Let $F_1 = f^{-1}(S_1)$. After a homotopy of f (fixed on ∂M), we may assume that F_1 is a 2-sided incompressible surface, no component of which is a 2-sphere. We may also assume that f maps $\mathcal{N}(F_1)$ onto $\mathcal{N}(S_1)$ in way that sends fibres homeomorphically to fibres. The following diagram commutes.

$$\begin{array}{ccc} F_1 & \xrightarrow{f} & S_1 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

By Theorem 3.3, $\pi_1(F_1) \to \pi_1(M)$ is injective and hence $\pi_1(F_1) \to \pi_1(S_1)$ is injective. Note that the restriction of f to ∂F_1 is a homeomorphism. If any component of F_1 is a disc, so is S_1, and hence so is every component of F_1. We therefore homotope $f|_{F_1}$ keeping $f|_{\partial F_1}$ fixed, so that it is a homeomorphism on each component. If no component of F_1 is a disc, then we may apply Theorem 12.1. Note that (ii) of Theorem 12.1 cannot hold, since $f|_{\partial F_1}$ is a homeomorphism. So we may homotope $f|_{F_1}$ to a covering map, keeping $f|_{\partial F_1}$ fixed. This homotopy extends to M, so that f still sends fibres of $\mathcal{N}(F_1)$ onto fibres of $\mathcal{N}(S_1)$. The cover $f|_{F_1}$ is a homeomorphism on its boundary, and hence is a homeomorphism. Therefore, f restricts to a map $M_2 = M - \mathrm{int}(\mathcal{N}(F_1)) \to N_2$ that is a homeomorphism between the boundaries of these 3-manifolds. Applying an argument similar to that in Theorem 3.3, we get that $\pi_1(M_2) \to \pi_1(M_1)$ is injective. Hence, $M_2 \to N_2$ is π_1-injective.

Arguing inductively, we may assume that (i), (ii) or (iii) holds for $M_2 \to N_2$. However, neither (ii) nor (iii) holds, except possibly $|p| = |s| = 1$ in (iii), since $f|_{\partial M_2}$ is a homeomorphism. Thus, $f|_{M_2}$ is a cover. It is a homeomorphism near ∂M_2, and therefore f is a homeomorphism. This proves the inductive step.

The induction starts with $M_n \to N_n$, with N_n a collection of 3-balls. Since the restriction of this map to each component of ∂M_n is π_1-injective, each component of ∂M_n is a 2-sphere. But M_n is irreducible. Hence, it is a collection of 3-balls. The map may therefore be homotoped to a homeomorphism.

Suppose now that N is closed. Let S be an orientable incompressible surface in N, no component of which is a 2-sphere. Then we may homotope f so that $F = f^{-1}(S)$ is an orientable incompressible surface in M, no component of which is a 2-sphere. As above, the map $f|_F \colon F \to S$ is π_1-injective and may therefore be homotoped to a cover. Also, $f|_{M-\mathrm{int}(\mathcal{N}(F))} \colon M - \mathrm{int}(\mathcal{N}(F)) \to N - \mathrm{int}(\mathcal{N}(S))$ is

π_1-injective. Apply the theorem in the case of bounded 3-manifolds to this map. No component of $M - \mathrm{int}(\mathcal{N}(F))$ satisfies (iii) of Theorem 12.3. If any component satisfies (ii), we may homotope f to reduce $|F|$. Therefore, we may assume that (i) holds for each component of $M - \mathrm{int}(\mathcal{N}(F))$. We have therefore homotoped f to a cover. $\square$

Three-dimensional manifolds
Michaelas Term 1999
Examples Sheet 1

Surfaces

1. Prove the 2-dimensional pl Schoëflies theorem: any properly embedded simple closed curve in a 2-sphere is ambient isotopic to the standard simple closed curve.

2. Classify (up to ambient isotopy) all the simple closed curves properly embedded in an annulus. What about such curves in a torus?

3. Show that any compact orientable surface with negative Euler characteristic is expressible as union of pairs of pants glued along their boundary curves.

4. Show that, if C is a homotopically trivial simple closed curve properly embedded in an orientable surface, then C bounds an embedded disc. (One approach to this is to use questions 2 and 3).

Surfaces in 3-manifolds

5. Show that if a prime orientable 3-manifold M contains a compressible torus boundary component, then M is the solid torus.

6. Find a compressible torus T properly embedded in some prime orientable 3-manifold M, such that no component of $M - T$ is a solid torus.

7. Let M be a compact 3-manifold. Suppose that we cut this 3-manifold along a sequence of properly embedded incompressible surfaces, and end with a collection of 3-balls. Show that M is prime. Apply this to the 3-manifold given as an example at the end of Lecture 1 (the space obtained by attaching thickened punctured tori to a thickened torus).

Heegaard splittings

8. Show that any closed orientable 3-manifold has Heegaard splittings of arbitrarily high genus.

9. Define the Heegaard genus $h(M)$ of a closed orientable 3-manifold M to be the minimal genus of a Heegaard splitting for M. Show that $h(M_1 \# M_2) \leq h(M_1) + h(M_2)$. (In fact, equality always holds.)

10. Find closed orientable 3-manifolds with arbitrarily large Heegaard genus.

DEHN SURGERY

11. Let L be a link in S^3. Let M be a 3-manifold obtained by surgery on L. Let C be a collection of simple closed curves, one on each component of $\mathcal{N}(L)$, that each bounds a disc in one of the attached solid tori, but none of which bounds a disc in $\partial\mathcal{N}(L)$. Show that the homeomorphism class of M only depends on the isotopy class of C in $\partial\mathcal{N}(L)$.

These curves C are usually specified by assigning a 'slope' in $\mathbb{Q} \cup \infty$ to each component of L. A slope p/q (where p and q are coprime integers) on a component K of L determines a curve on $\partial\mathcal{N}(K)$, which represents $(p, q) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(\partial\mathcal{N}(K))$. Here, the identification between $\mathbb{Z} \oplus \mathbb{Z}$ and $H_1(\partial\mathcal{N}(K))$ is chosen so that a curve representing $(1, 0)$ bounds a disc in $\partial\mathcal{N}(K)$ and a curve representing $(0, 1)$ is homologically trivial in $H_1(S^3 - K)$.

12. What is the manifold obtained by surgery on the unknot with slope 0? What about $1/q$ surgery, or more generally, p/q surgery on the unknot?

13. Show that any 3-manifold obtained by $1/q$ surgery on a knot in S^3 has the same homology as S^3.

14. Show that any 3-manifold M obtained by surgery on a knot, with slope zero, has $H_1(M) = \mathbb{Z}$. Construct an explicit non-separating orientable surface properly embedded in M.

15. Show that any closed orientable 3-manifold is obtained by surgery on a link in S^3 using only integral surgery slopes.

16. Construct a surgery descriptions of each lens space using only integral surgery slopes. (Express an element of $SL(2, \mathbb{Z})$ as a product of 'standard' matrices.)

17. Using question 12, show that any closed orientable 3-manifold is obtained by surgery on a link in S^3, where each component of the link is unknotted.

18. Is there a way of giving a surgery description of a compact orientable 3-manifold with non-empty boundary?

19. Let M be a 3-manifold obtained by surgery on the trefoil knot (the non-trivial knot with three crossings). Show that M has Heegaard genus at most two. (One of these spaces is the famous Poincaré homology 3-sphere.)

One-sided and two-sided surfaces

20. Show that if an orientable prime 3-manifold M contains a properly embedded $\mathbb{R}P^2$, then M is a copy of $\mathbb{R}P^3$.

21. In the lens space M obtained by $6/1$ surgery on the unknot, construct a properly embedded copy of the non-orientable surface N_3. Show that this is incompressible, but that the map $\pi_1(N_3) \to \pi_1(M)$ induced by inclusion is not injective.

Three-dimensional manifolds
Michaelmas Term 1999
Examples Sheet 2

1. Show that any compact non-orientable 3-manifold M having no $\mathbb{R}P^2$ boundary components has infinite $H_1(M)$, and hence has a 2-sided properly embedded non-separating incompressible surface.

2. Construct a Haken 3-manifold with the same homology as S^3.

3. Show that if M is a compact orientable irreducible 3-manifold with $\pi_1(M)$ a free group, then M is a handlebody. [Hint: consider a map from M to a bouquet of circles.]

4. Let M be a compact orientable irreducible 3-manifold and let F be a compact surface in ∂M. Show that if $\pi_1(F) \to \pi_1(M)$ is an isomorphism, then there is a homeomorphism from M to $F \times [0,1]$ taking F to $F \times \{0\}$.

5. Show that if S is an orientable incompressible surface properly embedded in a compact orientable 3-manifold M, then $\pi_1(M_S) \to \pi_1(M)$ is injective.

6. Using a hierarchy argument, show that the fundamental group of a Haken 3-manifold is torsion-free.

7. What extra assumptions do we need to make about a properly embedded incompressible surface with non-empty boundary in a compact irreducible 3-manifold to guarantee that it can be ambient isotoped into normal form?

8. Show that any Haken 3-manifold M has a hierarchy

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n,$$

where $n \leq 5$ (but where each surface S_i may be disconnected). Here, n is the *length* of this hierarchy. Which compact orientable 3-manifolds have a hierarchy of length one?

9. Suppose that a compact 3-manifold M contains k properly embedded 2-spheres, none of which bounds a 3-ball and no two of which are parallel. Then show that, for any triangulation of M, we may find such a collection of 2-spheres in normal form. Deduce that any compact 3-manifold can be

expressed as a connected sum of prime 3-manifolds.

10. Show that, in the statement of the Loop Theorem, we may remove the hypotheses that the 3-manifold is irreducible and compact.

11. Disprove the following conjecture. 'Let F be a surface properly embedded in a 3-manifold M, such that, with respect to any triangulation of M, F may be ambient isotoped into normal form. Then F is incompressible.' [Hint: let M be the lens space obtained by $6/1$ surgery on the unknot.]

12. Show that, for a fixed triangulation of a 3-manifold with t tetrahedra, its normal surfaces are in one-one correspondence with the integral lattice points in a subset C of $\mathbb{R}^{7t}$. If S_1, S_2 and S_3 are normal surfaces corresponding to points $[S_1]$, $[S_2]$ and $[S_3]$ in C, such that $[S_1] + [S_2] = [S_3]$, how are the Euler characteristics of S_1, S_2 and S_3 related?