Right exactness of tensor functor

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The functor $M \otimes_R -$ for R-modules is right exact, which is to say for any exact sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$, $M \otimes_R A \xrightarrow{\varphi_*} M \otimes_R B \xrightarrow{\psi_*} M \otimes_R C \to 0$ is also an exact sequence.

It is fairly straightforward to show directly on simple tensors that ψ_* is surjective and $\operatorname{im} \varphi_* \subset \ker \psi_*$, but the fact that $\operatorname{im} \varphi_* \supset \ker \psi_*$ is seemingly more mysterious: how is it that if $\varphi_*(\sum_i m_i \otimes b_i) = 0$ then $\sum_i m_i \otimes b_i$ is the image of something in $M \otimes_R A$?

One way to "see" this is using the adjunction between $- \otimes_R X$ and $\operatorname{Hom}_R(X, -)$ for any R-module X. Since $M \otimes_R -$ is a left adjoint functor, then it is right exact (since left adjoint functors preserve colimits, and in particular cokernels). In more detail, let P be an arbitrary R-module, then by applying $\operatorname{Hom}_R(-, P)$ to $A \to B \to C \to 0$ we get the left exact sequence

$$0 \to \operatorname{Hom}_R(C, P) \to \operatorname{Hom}_R(B, P) \to \operatorname{Hom}_R(A, P)$$

and by applying $\operatorname{Hom}_R(M, -)$ we get the left exact sequence

$$0 \to \operatorname{Hom}_R(M, \operatorname{Hom}_R(C, P)) \to \operatorname{Hom}_R(M, \operatorname{Hom}_R(B, P)) \to \operatorname{Hom}_R(M, \operatorname{Hom}_R(A, P))$$

and by applying the natural isomorphism $\operatorname{Hom}_R(M, \operatorname{Hom}_R(X, P)) \cong \operatorname{Hom}_R(M \otimes_R X, P)$, we get the left exact sequence

$$0 \to \operatorname{Hom}_R(M \otimes_R C, P) \to \operatorname{Hom}_R(M \otimes_R B, P) \to \operatorname{Hom}_R(M \otimes_R A, P)$$

Then we use a fact about the $\operatorname{Hom}_R(-, P)$ functor, which is that if the resulting sequence is exact for all P, then

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact as well. Let us prove this:

Lemma 1. *If* $\operatorname{Hom}_R(C, P) \xrightarrow{\psi^*} \operatorname{Hom}_R(B, P) \xrightarrow{\varphi^*} \operatorname{Hom}_R(A, P)$ *is exact for all R-modules P, then* $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ *is exact.*

Proof. With $P = C$, $\varphi^*(\psi^*(\mathrm{id}_C)) = \psi \circ \varphi$, which is zero by exactness, so $\operatorname{im} \varphi \subset \ker \psi$. With $P = \operatorname{coker} \varphi$ and $h : B \to \operatorname{coker} \varphi$, then $\varphi^*(h) = h \circ \varphi = 0$, so there is some $h' : C \to P$ with $h' \circ \psi = h$, which implies $\ker \psi \subset \ker h = \operatorname{im} \varphi$. $\square$

However, this is all fairly abstract. Can we see this more directly? The important part is that we want to show $\ker \psi_* = \operatorname{im} \varphi_*$, which is essentially saying we want to show that $M \otimes_R C$ is the cokernel of φ_*. Cokernels are the colimit of the diagram

$$\begin{array}{ccc} M \otimes_R A & \xrightarrow{\varphi_*} & M \otimes_R B \\ \big\downarrow{\scriptstyle 0} & & \\ 0 & & \end{array}$$

So, we want to show that whenever $g : M \otimes_R B \to N$ is a map for an R-module N, then there is a unique map $f : M \otimes_R C \to N$ making the following diagram commute:

$$\begin{tikzcd}
M \otimes_R A \arrow[r, "\varphi_*"] \arrow[d, "0"] & M \otimes_R B \arrow[d, "\psi_*"] \arrow[ddr, bend left, "g"] & \\
0 \arrow[r] \arrow[drr, bend right] & M \otimes_R C \arrow[dr, dotted, "\exists! f"] & \\
& & N
\end{tikzcd}$$

Let $f(m \otimes c) = g(m \otimes \psi^{-1}(c))$, where $\psi^{-1}(c)$ represents taking *any* preimage of c, and one exists since ψ is surjective. This is well-defined since when $b \in B$ is such that $\psi(b) = 0$, then there is some a with $\varphi(a) = b$, and then $g(m \otimes b) = g(m \otimes \varphi(a)) = g(\varphi_*(m \otimes a)) = 0$ by commutativity. Furthermore, $f(\psi_*(m \otimes b)) = f(m \otimes \psi(b)) = g(m \otimes b)$, so this f makes the diagram commutative. It is unique because ψ_* is surjective. Therefore, $M \otimes_R C$ is the cokernel of φ_*.

In particular, if $g : M \otimes_R B \to \operatorname{coker} \varphi_*$ is the quotient map, then $\psi_*(\sum_i m_i \otimes b_i) = 0$ implies $f(\psi_*(\sum_i m_i \otimes b_i)) = 0$ so $g(\sum_i m_i \otimes b_i) = 0$, and so $\sum_i m_i \otimes b_i \in \operatorname{im} \varphi_*$.

For sake of understanding this better, $\operatorname{coker} \varphi_*$ is $(M \otimes_R B)/\varphi_*(M \otimes_R A)$. The module $\varphi_*(M \otimes_R A)$ is $M \otimes_R \varphi(A)$ since $\varphi_*(m \otimes a) = m \otimes \varphi(a)$. There is an isomorphism $M \otimes_R B / M \otimes_R \varphi(A) \cong M \otimes_R (B/\varphi(A))$ by $[m \otimes b] \mapsto m \otimes [b]$, which is well-defined since for any $m \otimes x \in M \otimes_R \varphi(A)$, the image is $m \otimes [x] = m \otimes [0] = 0$. It has an inverse defined by $m \otimes [b] \mapsto [m \otimes b]$, similarly well-defined. Thus, $M \otimes_R A \to M \otimes_R B$ has a cokernel which is $M \otimes_R (B/\varphi(A))$, which is $M \otimes_R C$.