ALL THE WAYS I KNOW HOW TO DEFINE THE ALEXANDER POLYNOMIAL

KYLE A. MILLER

ABSTRACT. These began as notes for a talk given at the *Student 3-manifold seminar*, Spring 2019. There seems to be many ways to define the Alexander polynomial, all of which are somehow interrelated, but sometimes there is not an obvious path between any two given definitions. As the title suggests, this is an exploration of all the ways I know how to define this knot invariant. While we will touch on a number of facts and properties, these notes are not meant to be a complete survey of the Alexander polynomial.

CONTENTS

Date: May 3, 2019.

1. Introduction

Recall that a *link* is an embedded closed 1-manifold in S^3, and a *knot* is a 1-component link. As usual, our embeddings are either smooth or piecewise-linear.

In 1928, Alexander defined a polynomial invariant $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$, defined up to multiplication by $\pm t^n$, that is able to distinguish the knot types of all 35 knots with up to eight crossings [1]. His definition is purely an invariant of the group $\pi_1(S^3 - K)$.

There have been many ways the Alexander polynomial has been redefined, refined, and generalized, for example the Alexander-Conway polynomial, Reidemeister torsion, the HOMFLY-PT polynomial, knot Floer homology, Vassiliev invariants, and as a $U_q(\mathfrak{gl}(1|1))$ quantum invariant, to name a few. The plan is to go through as many of these as I can, hopefully in a not completely superficial manner.

No prior knowledge about the Alexander polynomial is assumed. The reader ought to be comfortable with first-year graduate algebraic topology and algebra, at least to get some basic definitions, and it might be helpful be familiar with a little 3-manifold topology. These notes might be updated from time to time.

2. Alexander's definition

This section is a summary of Alexander's original work from [1], with only mild refinements for modern sensibilities. Making sense of his procedure will be reserved for Section 3, but it might be worth knowing the quick overview: if $G = \pi_1(S^3 - K)$ for K a knot, then the commutator subgroup's abelianization is a module over the group ring of the abelianization of G, and the Alexander polynomial is the determinant of this module's presentation matrix, which is square. (Equivalently, we take the first homology of the infinite cyclic cover of $S^3 - K$ as a module over the group ring $\mathbb{Z}[H_1(S^3 - K)]$.)

2.1. The Dehn presentation.

Given a diagram for a knot K, Dehn described a procedure to write a presentation for $\pi_1(S^3 - K)$. Choose a basepoint well outside the diagram. Each face corresponds to a group generator by taking a loop that goes from the basepoint, over the plane, down through the face, then back under the plane to the basepoint (Figure 1).

Figure 1. A generator for the Dehn presentation fig:dehn-generator

For the "outer" face, where the basepoint lies, the corresponding generator is trivial. The other relations come from crossings.

Imagine a nullhomotopic loop sitting between the two strands of a crossing, oriented counterclockwise in Figure 2. This can be read off as the word $x_i x_j^{-1} x_k x_\ell^{-1}$. If m is the "outer" face, then

$$\pi_1(S^3 - K) \cong \langle x_1, \dots, x_m \mid x_m = 1 \text{ and all crossing relations} \rangle.$$

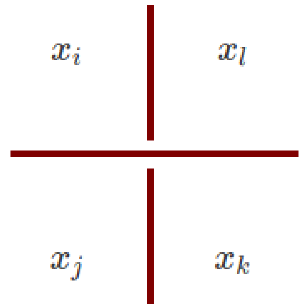


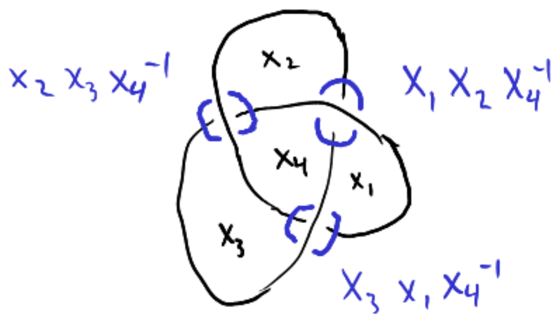
FIGURE 2. A crossing with the regions labeled by their Dehn generators. This crossing induces the relation $x_i x_j^{-1} x_k x_\ell^{-1} = 1$.

fig:dehn-relation

This can be proved using the van Kampen theorem: cutting out a ball neighborhood from each crossing gives a space that is the complement of a planar 4-valent graph (whose fundamental group is free since its complement is a handlebody), and rather than gluing the balls back in, one can get a homotopy equivalent space by gluing in disks that separate the two strands at a crossing.

example:dehn-trefoil

Example 2.1. For the trefoil knot 3_1, we take the usual diagram and label all the "interior" faces with generators then work out the relations at the crossings:



Since it is easy enough to immediately eliminate the generator for the "outer" face, we do so. Hence,

$$\pi_1(S^3 - 3_1) \cong \langle x_1, x_2, x_3, x_4 \mid x_1 x_2 x_4^{-1}, x_2 x_3 x_4^{-1}, x_3 x_1 x_4^{-1} \rangle.$$

Remark 2.2. This is the presentation Dehn used in 1914 to show that the outer automorphism group $\operatorname{Out} \pi_1(S^3 - 3_1)$ (of automorphisms modulo inner automorphisms) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, from which fact one can show that the trefoil knot is chiral [8].

Remark 2.3. Notice that $x_4 = x_3 x_1$, so we can eliminate the generator to get

$$\pi_1(S^3 - 3_1) \cong \langle x_1, x_2, x_3 \mid x_1 x_2 x_1^{-1} x_3^{-1}, x_2 x_3 x_1^{-1} x_3^{-1} \rangle.$$

This reveals $x_3 = x_1 x_2 x_1^{-1}$, so with one more elimination we get

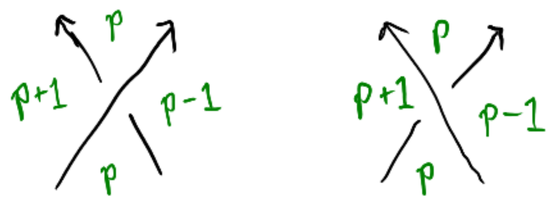
$$\pi_1(S^3 - 3_1) \cong \langle x_1, x_2 \mid x_2 x_1 x_2 x_1^{-1} x_2^{-1} x_1^{-1} \rangle$$

This is the three-strand braid group B_3 (the fundamental group of the unordered configuration space of three distinct points in $\mathbb{C}$).

2.2. **Abelianization.** Recall that the abelianization $H_1(G)$ of a group G is the universal group that every homomorphism $G \to A$ factors through, for A an abelian group. Concretely, $H_1(G) \cong G/[G,G]$ where $[G,G]$ is the commutator subgroup.

Also, recall that $H_1(\pi_1(S^3 - K)) \cong H_1(S^3 - K) \cong \mathbb{Z}$. The first isomorphism is the Hurewicz theorem, and the second isomorphism is due to Alexander duality.

The abelianization of the Dehn presentation can be described using winding numbers. After giving K an orientation, the homomorphism $\pi_1(S^3 - K) \to \mathbb{Z}$ is from assigning to each generator x_i the winding number of K about a point in its corresponding region, or equivalently from the linking number between x_i and K. This is reflected in the following relations, where the green numerical label is the image of that region's generator in $\mathbb{Z}$:



Example 2.4. Here is an abelianization for the trefoil's group from Example 2.1:

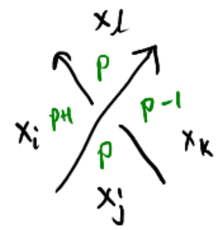
2.3. **The associated matrix.** Alexander takes the Dehn presentation and puts it into "canonical form," and from there he produces the *associated matrix*. If $x_0, x_1, \dots, x_n, x_{n+1}$ are the generators for every region, with $x_0 = 1$ corresponding to the "outer" region and x_{n+1} corresponding to a region neighboring the one for x_0, then for $p : \pi_1(S^3 - K) \to \mathbb{Z}$ being the abelianization one can substitute

$$\begin{aligned} s &= x'_{n+1} = x_{n+1} \\ x'_1 &= s^{-p(x_1)} x_1 \\ &\vdots \\ x'_n &= s^{-p(x_n)} x_n \end{aligned}$$

to get a system of generators $s, x'_1, \dots, x'_n$ such that $p(x'_i) = 0$ for $1 \leq i \leq n$ and $p(s) = \pm 1$. Substituting these into the relation $x_i x_j^{-1} x_k x_\ell^{-1} = 1$, where x_i and x_j are the regions to the left of the understrand (with respect to the strand's orientation), gives a relation of the form

$$(s x'_i s^{-1})\,(s (x'_j)^{-1} s^{-1})\, x'_k\, (x'_\ell)^{-1} = 1.$$

For example, in the following crossing with the labeled abelianization values,



we have

$$x_i x_j^{-1} x_k x_\ell^{-1} = (s^{p+1} x_i')\,((x_j')^{-1} s^{-p})\,(s^{p-1} x_k')\,((x_\ell')^{-1} s^{-p})$$
$$= s^p\,(s x_i' s^{-1})\,(s(x_j')^{-1} s^{-1})\,x_k'\,(x_\ell')^{-1}\,s^{-p},$$

and then we can conjugate the relation by s^{-p}.

From here, Alexander writes x^t for sxs^{-1} and then formally linearizes the relations to get equations of the form

$$tx_i' - tx_j' + x_k' - x_\ell' = 0.$$

The *associated matrix* M is a matrix whose entry M_{ij} is the coefficient in front of x_j' for the formally linearized relation from crossing i. Alexander's mnemonic is illustrated in Figure 3—the two regions to the left of the understrand are marked with a dot, and in counterclockwise order the regions are taken as an alternating sum with the dotted regions multiplied by t.

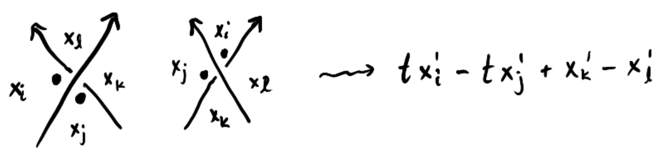


Figure 3. Mnemonic for the associated matrix fig:alexander-mnemonic

example:ass-mat-trefoil

Example 2.5. From Example 2.1, we get abelianized relations

$$0 = tx_2' - tx_4' + x_1'$$
$$0 = tx_3' - tx_4' + x_2'$$
$$0 = tx_1' - tx_4' + x_3',$$

using that the "outer" region is zero. These give the associated matrix

$$\begin{bmatrix} 1 & t & 0 & -t \\ 0 & 1 & t & -t \\ t & 0 & 1 & -t \end{bmatrix}.$$

def:original-alexander-poly

Definition 2.6. Let M be an associated matrix for a knot K, and let M' be the result of crossing off the s column of M (that is, cross off any column associated to a region neighboring the "outer" region). The *Alexander polynomial* $\Delta_K(t)$ is the determinant of M'.

We will defer a proof that this is well-defined until Theorem 3.25.

Example 2.7. Any of the first three columns of the associated matrix in Example 2.5 is "the s column," so we calculate that

$$\Delta_{3_1}(t) = \det \begin{bmatrix} 1 & t & -t \\ 0 & 1 & -t \\ t & 0 & -t \end{bmatrix} = -t(1-t+t^2).$$

Recall that this is only defined up to multiplication by $\pm t^n$.

Remark 2.8. I slightly modified Alexander's definition. In the original, the "outer" face still has a column, and then two columns with adjacent abelianization values must be removed before the determinant is computed.

3. The Alexander modules

sec:alexander-modules

According to Rolfsen [24], Alexander knew that this rather *ad hoc* definition comes from the following (explained by Milnor in [22] for chains over a field). Let X be a topological space with $G = \pi_1(X)$. The abelianization $\pi_1(X) \to H_1(X)$ corresponds to the *universal abelian cover* $p : \overline{X} \to X$, where $\operatorname{im} p_* = [G, G]$. The group $H_1(X)$ acts on $\overline{X}$ by deck transformations, and hence it acts on the chain complex $C_\bullet(\overline{X})$.

Definition 3.1. The $\mathbb{Z}[H_1(X)]$-module $H_i(\overline{X})$ is the ith *Alexander module* for X.

In the case of $X = S^3 - K$, the space $\overline{X}$ is known as the *infinite cyclic cover*. The group algebra $\mathbb{Z}[H_1(X)]$ is the ring of Laurent polynomials $\mathbb{Z}[t^{\pm 1}]$ where t is a generator in $H_1(X)$, the image of a meridian. Most of the Alexander modules of knot complements are trivial:

- $H_0(\overline{X}) = \mathbb{Z}[t^{\pm 1}]/(t-1) \cong \mathbb{Z}$ since t acts by the identity.
- $H_i(\overline{X}) = 0$ if $i \geq 2$. Since $\overline{X}$ is a noncompact 3-manifold, this is clear for $i \geq 3$. For $i = 2$, one can show this by decomposing $\overline{X}$ using lifts of a Seifert surface then carefully applying the Mayer-Vietoris sequence. A somewhat fancy way of showing this fact is as follows (from [22] and [29]). Consider the short exact sequence of chain complexes

  $$0 \to C_i(\overline{X}; \mathbb{Q}) \xrightarrow{t-1} C_i(\overline{X}; \mathbb{Q}) \xrightarrow{p_\sharp} C_i(X; \mathbb{Q}) \to 0,$$

  which can come from thinking about $C_i(\overline{X}; \mathbb{Q})$ as $\mathbb{Q}[t^{\pm 1}] \otimes C_i(X)$ after choosing lifts of each chain then noting that $p_\sharp$ is from $t \mapsto 1$. This gives rise to a long exact sequence

  $$\cdots \to H_{i+1}(X; \mathbb{Q}) \xrightarrow{\partial} H_i(\overline{X}; \mathbb{Q}) \xrightarrow{t-1} H_i(\overline{X}; \mathbb{Q}) \xrightarrow{p_*} H_i(X; \mathbb{Q}) \to \cdots.$$

  Since $H_j(X; \mathbb{Q}) = 0$ for $i \geq 2$, we have an isomorphism

  $$H_2(\overline{X}; \mathbb{Q}) \xrightarrow{t-1} H_2(\overline{X}; \mathbb{Q}).$$

  The space X deformation retracts onto a 2-skeleton, so $\overline{X}$ is also two-dimensional. This means

  $$H_2(\overline{X}; \mathbb{Q}) \cong \ker(C_2(\overline{X}; \mathbb{Q}) \xrightarrow{\partial} C_1(\overline{X}; \mathbb{Q})),$$

  so, since $\mathbb{Q}[t^{\pm 1}]$ is a PID, $H_2(\overline{X}; \mathbb{Q})$ is a finitely generated free $\mathbb{Q}[t^{\pm 1}]$-module. Because $0 \to \mathbb{Q}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Q}[t^{\pm 1}] \to \mathbb{Q} \to 0$ is exact, it follows that $t-1$ cannot be surjective, and thus $H_2(\overline{X}; \mathbb{Q}) = 0$. Similarly, $H_2(\overline{X}; \mathbb{Z})$ is a free $\mathbb{Z}$-module due to $\overline{X}$ being two-dimensional. By the universal coefficient theorem, $H_2(\overline{X};\mathbb{Q}) = \mathbb{Q}[t^{\pm 1}] \otimes H_2(\overline{X};\mathbb{Z})$, and therefore $H_2(\overline{X};\mathbb{Z}) = 0$.

Hence the only interesting Alexander module for a knot is $H_1(\overline{X})$, sometimes called *the* Alexander module of a knot.

Remark 3.2. The above long exact sequence ends (over $\mathbb{Z}$) with

$$\begin{array}{ccccccccccc}
H_1(\overline{X}) & \xrightarrow{t-1} & H_1(\overline{X}) & \xrightarrow{p_*} & H_1(X) & \xrightarrow{\partial} & H_0(\overline{X}) & \xrightarrow{t-1} & H_0(\overline{X}) & \xrightarrow{p_*} & H_0(X) \\
 & & & & \| & & \| & & \| & & \| \\
 & & & & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}$$

Thus $H_1(\overline{X}) \xrightarrow{t-1} H_1(\overline{X})$ is surjective, and since $H_1(\overline{X})$ is finitely generated as a $\mathbb{Z}[t^{\pm 1}]$-module, $t-1$ is injective as well.

[This is actually a rather neat fact from ring theory due to Vasconcelos: if R is a commutative ring, M is a finitely generated R-module, and $f : M \to M$ is a surjective homomorphism, then f is an isomorphism.

Take M as an $R[f]$-module. The ideal $I = (f)$ has $IM = M$ by surjectivity. Nakayama's lemma is that $IM = M$ implies there is an $i \in I$ such that $im = m$ for all $m \in M$ (the lemma only uses that M is finitely generated). Since $i = pf$ for some $p \in R[f]$, $fm = 0$ implies $m = im = pfm = 0$, so f is injective!]

Remark 3.3. For general X, the first Alexander module only depends on $G = \pi_1(X)$ and the choice of abelianization. Let $G \rhd G' \rhd G'' \rhd \cdots$ be the derived series for G, where $G' = [G,G]$, $G'' = [G',G']$, and so on.

G acts on each derived subgroup by conjugation, so the action descends to quotients $G^{(i)}/G^{(i+1)}$ for all $i \geq 0$. The kernel of this action is all those $g \in G$ such that for all $h \in G^{(i)}$, $ghg^{-1}G^{(i+1)} = hG^{(i+1)}$, which by normality is the condition $[g,h] \in G^{(i+1)}$. Hence the abelian group $G^{(i)}/G^{(i+1)}$ is a $\mathbb{Z}[G/G^{(i)}]$-module. Cochran calls these *higher-order Alexander modules* [3], which are a source of "nonabelian" invariants.

One can identify $H_1(\overline{X})$ with G'/G'' since $\pi_1(\overline{X}) = G'$, and through the above discussion we see how G'/G'' is a (G/G')-module in purely group-theoretic terms.

If G is a finitely presented group, then so is G/G', and furthermore G'/G'' is a finitely presented $\mathbb{Z}[G/G']$-module. This fact will be made clear in Section 3.3.

Remark 3.4. A *metabelian* group G is a group whose commutator subgroup is abelian (that is, one whose derived series terminates at $G'' = 1$). The first Alexander module of a group is a "metabelian" invariant, meaning G/G'' has the same first Alexander module as that of G.

The idea of the Alexander polynomial is to capture some of the essence of $H_1(\overline{X})$ in a way that can be put into a normal form. The classification of finitely presented $\mathbb{Z}[t^{\pm 1}]$-modules is not so straightforward!

3.1. **Orders.** While the ring $\mathbb{Z}[t^{\pm 1}]$ is a Noetherian unique factorization domain, it is not a principal ideal domain, so the structure theorem does not apply to $H_1(\overline{X})$. Passing to $\mathbb{Q}$ coefficients, we have $\mathbb{Q}[t^{\pm 1}]$, which *is* a PID.

In general, for R a PID and M a finitely generated R-module, the structure theorem gives a decomposition into cyclic modules

$$M \approx \bigoplus_{i=1}^{n} R/(x_i)$$

for some elements $x_i \in R$ (possibly zero). The *order* of M is the ideal $\operatorname{order}(M) = (\prod_{i=1}^{n} x_i)$, which is a well-defined invariant of the module.

Remark 3.5. As it turns out, $\Delta_K(t)$ generates $\operatorname{order}(H_1(\overline{X};\mathbb{Q}))$. The order can recover $\Delta_K(t)$ by using Corollary 3.40, that $\Delta_K(1) = \pm 1$.

3.2. **Elementary ideals.** There is a way to fix order theory for the failure of $\mathbb{Z}[t^{\pm 1}]$ being a principal ideal domain that instead uses its being a unique factorization domain.

For an R-module M, a *free presentation* an exact sequence

$$F \xrightarrow{A} G \to M \to 0$$

with F and G both free R-modules (so $M \cong \operatorname{coker} A$). By choosing bases for F and G, we can regard A as a *presentation matrix*.

Definition 3.6. Let M be a R-module with a presentation $F \xrightarrow{A} R^s \to M \to 0$ for some $s \in \mathbb{N}$. The ith *elementary ideal* (or *Fitting ideal*) $\mathcal{E}_i(M) \subset R$ is the ideal generated by the $(s-i)\times(s-i)$ minors of the presentation matrix A. Equivalently, it is the ideal generated by the image of the map $\wedge^{s-i} F \otimes \wedge^{s-i}(R^s)^* \to R$ induced by $\wedge^{s-i} A : \wedge^{s-i} F \to \wedge^{s-i} G$.

That this definition does not depend on the presentation for M is a matter of checking that the elementary ideals remain the same after elementary transformations of the presentation matrix. The general case is given in [9, Corollary 20.4], and the case of finitely presented modules (where F is finite rank) can be seen in [7, 11, 20].

The first Alexander module is finitely presented as a $\mathbb{Z}[t^{\pm 1}]$-module, which we will see in Section 3.3.

Definition 3.7. The ith *Alexander polynomial* $\Delta_K^i(t)$ of a knot K is the GCD of the ideal $\mathcal{E}_i(H_1(\overline{S^3 - K}))$. (Recall: the GCD of an ideal is the smallest principal ideal containing it.)

As it will turn out in Theorem 3.25, $\Delta_K(t) = \Delta_K^0(t)$ is *the* Alexander polynomial

Remark 3.8. Since determinants have cofactor expansions, the elementary ideals of an R-module M form a filtration

$$\mathcal{E}_0(M) \subseteq \mathcal{E}_1(M) \subseteq \cdots \subseteq \mathcal{E}_s(M) = R.$$

The GCDs of these ideals form a parallel filtration. Thus, each Alexander polynomial of a knot is divisible by the next. The infinite sequence

$$(\Delta_K^0/\Delta_K^1, \Delta_K^1/\Delta_K^2, \dots, \Delta_K^{s-1}/\Delta_K^s, \Delta_K^s, 1, 1, \dots)$$

is an ordered factorization of the Alexander polynomial that is an invariant of K.

Example 3.9. Both 8_{18} and 9_{24} have $\Delta_K^0(t) = (1-3t+t^2)(1-t+t^2)^2$, but $\Delta_{8_{18}}^1(t) = t^2 - t + 1$ and $\Delta_{9_{24}}^1(t) = 1$. Their modules are $H_1(\overline{S^3 - 8_{18}}) \cong \mathbb{Z}[t^{\pm 1}]/((1-3t+t^2)(1-t+t^2)) \oplus \mathbb{Z}[t^{\pm 1}]/(1-t+t^2)$ and $H_1(\overline{S^3 - 9_{24}}) \cong \mathbb{Z}[t^{\pm 1}]/((1-3t+t^2)(1-t+t^2)^2)$.

Example 3.10. The knot 6_1 has Alexander module $\mathbb{Z}[t^{\pm 1}]/(2-5t+2t^2)$ and the knot 9_{46} has Alexander module $\mathbb{Z}[t^{\pm 1}]/(2-t) \oplus \mathbb{Z}[t^{\pm 1}]/(2t-1)$, so they are distinguishable by their modules but not by their Alexander polynomials (in fact, $\mathcal{E}_0$ is the same for both but $\mathcal{E}_1$ is different: (1) versus $(3, 1+t)$). Over $\mathbb{Q}$, $H_1(\overline{S^3 - 6_1}; \mathbb{Q}) \cong H_1(\overline{S^3 - 9_{46}}; \mathbb{Q})$. This is noted in Gordon's survey [14].

Remark 3.11. The formation of elementary ideals commutes with base change [9, Corollary 20.5]: for a ring homomorphism $R \to S$ and an R-module M,

$$\mathcal{E}_j(S \otimes_R M) = S\mathcal{E}_j(M).$$

For example, $\mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{Q}[t^{\pm 1}]$ gives

$$\mathcal{E}_j(H_1(\overline{S^3 - K}; \mathbb{Q})) = \mathbb{Q}[t^{\pm 1}]\mathcal{E}_j(H_1(\overline{S^3 - K})).$$

What can happen is that $\mathbb{Z}[t^{\pm 1}]$ ideals of the form $(p, q(t))$ for $p \in \mathbb{Z} - \{0\}$ and $q(t) \in \mathbb{Z}[t^{\pm}]$ become $\mathbb{Q}[t^{\pm 1}]$.

Proposition 3.12 (Crowell [6], [9, Exercise 20.6])**.** *The annihilator* $\operatorname{Ann}_{\mathbb{Z}[t^{\pm 1}]} H_1(\overline{S^3 - K})$ *of the Alexander module for a knot K is the principal ideal generated by* $\Delta_K^0(t)/\Delta_K^1(t)$.

Proof. Consider an $n \times n$ square presentation matrix $A : \mathbb{Z}[t^{\pm 1}]^n \to \mathbb{Z}[t^{\pm 1}]^n$ for the Alexander module of a knot K (Theorem 3.38). There is an adjugate/cofactor matrix A^* satisfying $AA^* = A^*A = \det(A)I = \Delta_K^0(t)I$, and the entries of A^* are all the $(n-1) \times (n-1)$ minors of A. Since $\Delta_K^1(t)$ is the GCD of the entries of A^*, letting $\mu_0 = \Delta_K^0(t)/\Delta_K^1(t)$ we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[t^{\pm 1}]^n & \xrightarrow{A} & \mathbb{Z}[t^{\pm 1}]^n \\ {\scriptstyle A^*/\Delta_K^1(t)}\downarrow & \searrow^{\mu_0 I} & \downarrow{\scriptstyle A^*/\Delta_K^1(t)} \\ \mathbb{Z}[t^{\pm 1}]^n & \xrightarrow{A} & \mathbb{Z}[t^{\pm 1}]^n. \end{array}$$

An element $f \in \mathbb{Z}[t^{\pm 1}]$ is in the annihilator iff $f\mathbb{Z}[t^{\pm 1}]^n \subset \operatorname{im} A$. By the upper triangle of the diagram, for f in the annihilator, $f \operatorname{im}(A^*/\Delta_K^1(t)) \subset \mu_0 \mathbb{Z}[t^{\pm 1}]^n$, and since the entries of $A^*/\Delta_K^1(t)$ are coprime, we see $f \in (\mu_0)$. By the lower triangle of the diagram, multiplying an element by μ_0 puts it into the image of A, so μ_0 is in the annihilator. Therefore, the annihilator is the principal ideal (μ_0). □

Example 3.13. If $\Delta_K^0(t) = 1$, then $H_1(\overline{S^3 - K}) = 0$, which implies the derived series stabilizes at $G' = G''$. (G' is a *perfect* group.)

Example 3.14. Extending the ring to the PID $\mathbb{C}[t^{\pm 1}]$, we have a decomposition

$$H_1(\overline{S^3 - K}; \mathbb{C}) = \bigoplus_{i=1}^{n} \mathbb{C}[t]/(p_i(t))$$

where $p_i(t) \in \mathbb{C}[t]$ and, for $i > 1$, $p_{i-1}(t) \mid p_i(t)$. There is a diagonal presentation matrix with $p_i(t)$ at position (i,i). $\mathcal{E}_k(H_1(\overline{S^3 - K}; \mathbb{C}))$ is generated by the product $p_1(t) \cdots p_{n-k}(t)$. The annihilator is generated by $p_n(t)$.

Remark 3.15. For an R-module M, the closed subset of $\operatorname{Spec} R$ defined by $\mathcal{E}_i(M)$ defines the set of prime ideals Q for which the localization M_Q cannot be generated by i elements [9, Proposition 20.6]. Less precisely, if $\mathcal{E}_i(M) \neq R$ then M cannot be generated by i elements.

For $i > 0$, $\operatorname{Ann}(M)\mathcal{E}_j(M) \subset \mathcal{E}_{j-1}(M)$, and if M can be generated by n elements, $\operatorname{Ann}(M)^n \subset \mathcal{E}_0(M)$ [9, Proposition 20.6].

Remark 3.16. By looking at the effect of ±1 surgery on crossings:

Theorem 3.17 ([20, Theorem 7.10]). *If, for K a knot, $\mathcal{E}_i(H_1(\overline{S^3 - K})) \neq \mathbb{Z}[t^{\pm 1}]$, then K has unknotting number $u(K) > i$.*

Remark 3.18. Unlike the case over $\mathbb{Q}$, the first Alexander module might not be a direct sum of cyclic modules. The pretzel knots $K_1 = P(107,-1,3)$ and $K_2 = P(25,-3,13)$ have the same elementary ideals, but $H_1(\overline{S^3 - K_1})$ is a direct sum of cyclic modules where $H_1(\overline{S^3 - K_2})$ is not. This uses an invariant of the module called the *row class* [12].

The following theorem gives some structure for the Alexander module for certain knots. (We will see that Alexander polynomials of knots are always of the form specified in the theorem.)

thm:crowell-zmod-struct

Theorem 3.19 (Crowell [5]). *If K is a knot with knot group $G = \pi_1(S^3 - K)$ and Alexander polynomial*

$$\Delta_K(t) = c_0 + c_1 t + \cdots + c_{h-1}t^{h-1} + c_h t^h + c_{h-1}t^{h+1} + \cdots + c_1 t^{2h-1} + c_0 t^{2h},$$

and if $c_0 = p_1^{k_1} \dots p_s^{k_s}$ is a prime decomposition, then as $\mathbb{Z}$-modules

$$H_1(\overline{S^3 - K}) = G'/G'' = \bigoplus_{i=1}^{2h} \mathbb{Z}[c_0^{\pm 1}]$$

iff $c_j \equiv 0 \pmod{p_i}$ for all $i = 1, \dots, s$ and $j = 0, \dots, h-1$.

Remark 3.20. There is a structure theorem for finitely generated $\mathbb{Z}[t,t^{-1}]$-modules M with no $\mathbb{Z}$-torsion (such as the first Alexander module of a knot complement) in [26]. There exists a pair U, B of finitely generated abelian groups and monomorphisms $f, g : U \to B$ such that M is isomorphic to the infinite fibered coproduct

$$\cdots \oplus_U B \oplus_U B \oplus_U \cdots$$

with identical amalgamations $B \xleftarrow{g} U \xrightarrow{f} B$, and $\langle t \rangle$ acts on this by shifting coordinates to the left or right. If U is generated by q elements, then $\deg \Delta(M) \leq q$. If M is the first Alexander module of a knot complement and $\Delta_K(t)$ has breadth d, then $U, B = \mathbb{Z}^d$ and $\Delta_K(t) = \det(tg - f)$. Compare this to Section 3.8, where $U = H_1(\Sigma)$ for a Seifert surface Σ and $B = H_1(S^3 - \Sigma)$; then f, g are the induced maps from both pushoffs of Σ into $S^3 - \Sigma$.

sec:fox-calc

3.3. **The Fox calculus.** In [11], Fox defines a way to compute the presentation matrix for $H_1(\overline{X})$. In particular, the Fox free calculus gives a way to compute the entries of a matrix for the chain map ∂_2 for a finite CW chain complex whose homology gives $H_1(\overline{X})$. For a section called "The Fox calculus," we will be discussing his actual calculus very little: it turns out that once one understands a good chain complex for $\overline{X}$ that there is a somewhat more direct method to calculate this ∂_2 matrix. Nevertheless, at some point we will mention Fox derivatives.

Since $H_1(\overline{X})$ depends only on $G = \pi_1(X)$, we may as well use a *presentation complex* for $G = \langle g_1,\dots,g_s \mid R_1,\dots,R_r\rangle$. Recall that the complex X_G is given by

- A single 0-cell known as $*$.
- A 1-cell g_i for each generator g_i.
- A 2-cell for each relator R_j, where the attachment map is given by the sequence of generators in the word.

Let $f : G \to \langle t\rangle$ be the abelianization. To construct $C_i(\overline{X_G})$, we can use the fact that all lifts of cells in X_G to $\overline{X_G}$ are related by the $\langle t\rangle$ action. Hence, we may identify the cells of X_G with some distinguished lifts to $\overline{X_G}$:

- Let $*$ be any lift.
- Let g_i denote the lift that starts from $*$.
- Let R_j denote the lift where the boundary word starts from $*$.

This gives an explicit isomorphism $C_i(\overline{X_G}) \cong \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}} C_i(X_G)$ as $\mathbb{Z}$-modules. In particular,

- $C_0(\overline{X_G}) = \mathbb{Z}[t^{\pm 1}]\langle *\rangle$,
- $C_1(\overline{X_G}) = \mathbb{Z}[t^{\pm 1}]\langle g_1,\dots,g_s\rangle$, and
- $C_2(\overline{X_G}) = \mathbb{Z}[t^{\pm 1}]\langle R_1,\dots,R_r\rangle$.

Now to determine the boundary maps. The first is $\partial_1(g_i) = (f(g_i) - 1)*$ since the endpoint of g_i in the lift is its value in the abelianization, by definition of $\overline{X_G}$.

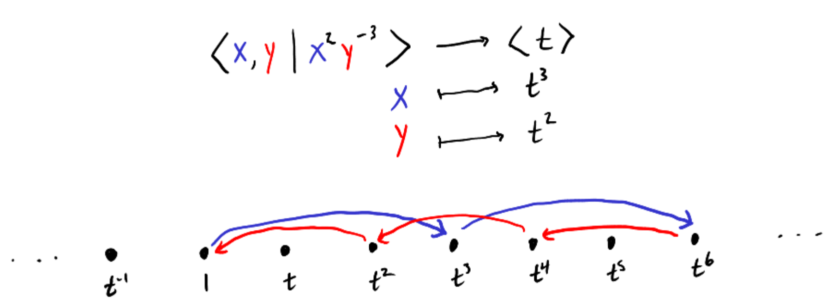
The second boundary map is a bit more involved. In the lift of R_j, we have to take into account how each generator in its boundary must lift end-to-end in $\overline{X_G}$. These translations are determined by the abelianization of each prefix. In particular, if $R_j = g_{j_1}^{\epsilon_1}\cdots g_{j_k}^{\epsilon_k}$, where $\epsilon_\ell = \pm 1$ for each ℓ, then

$$\partial_2(R_j) = \sum_{\ell=1}^{k} f(g_{j_1}^{\epsilon_1}\cdots g_{j_{\ell-1}}^{\epsilon_{\ell-1}})g_{j_\ell}^{\epsilon_\ell},$$

where in $C_1(\overline{X_G})$ we interpret g_i^{-1} to mean $-f(g_i)g_i$, since $-g_i$ has the opposite orientation from that of g_i and $f(g_i)$ shifts the 1-chain so it "begins" at $*$.

ex:trefoil-torus

Example 3.21. The following illustrates the boundary of the relator in $C_1(\overline{X_G})$ for the group $G = \langle x, y \mid x^2 = y^3\rangle$.



Algebraically, $\partial_2(x^2y^{-3}) = (1+t^3)x-(1+t^2+t^4)y$. We also have $\partial_1 x = t^3-1$ and $\partial_1 y = t^2-1$, hence the group has the corresponding chain complex

$$\mathbb{Z}[t^{\pm 1}] \xrightarrow{\begin{bmatrix} 1+t^3 \\ -(1+t^2+t^4) \end{bmatrix}} \mathbb{Z}[t^{\pm 1}]^2 \xrightarrow{\begin{bmatrix} t^3-1 & t^2-1 \end{bmatrix}} \mathbb{Z}[t^{\pm 1}].$$

The kernel of ∂_1 is those vectors (a,b) for which $a(t^3-1)+b(t^2-1)=0$. We can factor out $t-1$, so equivalently the equation is $a(t^2+t+1)+b(t+1)=0$. As these polynomials are coprime, $t+1 \mid a$ and $t^2+t+1 \mid b$, thus $a=(t+1)c$ and $b=(t^2+t+1)c$ for $c \in \mathbb{Z}[t^{\pm 1}]$. Hence, $\ker\partial_1$ is generated by $(t+1, t^2+t+1)$. The image of ∂_2 is generated by $((t+1)(t^2-t+1), (t^2+t+1)(t^2-t+1))$. Therefore $H_1(\overline{X_G}) = \mathbb{Z}[t^{\pm 1}]/(t^2-t+1)$.

Remark 3.22. The group in the preceding example is equivalent to that of the trefoil. Thus $\Delta_{3_1}(t) = t^2-t+1$, like before.

Proposition 3.23. *Extending ∂_2 to words in G in general, then*

(1) $\partial_2(1) = 0$;
(2) *if g_i is a generator,* $\partial_2(g_i) = g_i$; *and*
(3) *if w_1 and w_2 are two words in G, then* $\partial_2(w_1w_2) = \partial_2(w_1) + f(w_1)\partial_2(w_2)$.

These define ∂_2. A corollary is that $\partial_2(w^{-1}) = -f(w)\partial_2(w)$ for every word w in G.

We do not especially need the following definition, but we have it here for completeness. With it, the boundary map renders as $\partial_2(R_i) = \sum_{j=1}^{s} f(\frac{\partial R_i}{\partial g_j})g_j$, where f is extended to $\mathbb{Z}[G] \to \mathbb{Z}[t^{\pm 1}]$. This allows one to calculate a particular entry of the matrix for ∂_2 if one so wishes. One can imagine the relationship as between total derivatives and partial derivatives.

Definition 3.24. Let $F_n = \langle g_1, \dots g_n \rangle$ be a free group on n generators. The *Fox derivative* is a $\mathbb{Z}$-module map $\frac{\partial}{\partial g_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ characterized by

- $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$, and
- $\frac{\partial w_1 w_2}{\partial x_i} = \frac{\partial w_1}{\partial x_i}\varepsilon(w_2) + w_1\frac{\partial w_2}{\partial x_i}$ for $w_1, w_2 \in \mathbb{Z}[F_n]$ and $\varepsilon(\sum_i n_i g_i) = \sum_i n_i$.

thm:alex-poly-is-elementary

Theorem 3.25. *The $\Delta_K^0(t)$ definition of the Alexander polynomial is equivalent to Alexander's. Furthermore, $\mathcal{E}_0(H_1(\overline{S^3-K}))$ is a principal ideal.*

Proof. Recall the transformation of the Dehn presentation into "canonical form." The effect of this was that in the abelianization, $s \mapsto t$ and $x_1', \dots, x_n' \mapsto 1$. We have $\partial_2(sx_i's^{-1}) = s + tx_i' - s = tx_i'$ and $\partial_2(s(x_i')^{-1}s^{-1}) = -tx_i'$, so the "formal linearization" of a relation R_j in canonical form is $\partial_2 R_j$:

$$tx_i' - tx_j' + x_k' - x_\ell' = 0.$$

In particular, the "associated matrix" is the matrix for ∂_2. The matrix for ∂_1 is given by $\partial_1 s = t-1$ and $\partial_1 x_i' = 0$, so the kernel is given by all chains with no s term. Hence, the presentation matrix for $H_1(\overline{X})$ is the result of removing the s column of the associated matrix. Since the resulting matrix is square, the GCD of $\mathcal{E}_0$ is simply the determinant of this matrix. □

3.4. **Torus knots.** The calculation from Example 3.21 can be generalized to all torus knots. A (p,q) *torus knot* $T_{p,q}$ is, with p,q coprime, the curve with p/q slope on the boundary of a tubular neighborhood of an unknot. As an application of the van Kampen theorem, [15, Example 1.24] gives the presentation

$$\pi_1(S^3 - T_{p,q}) = \langle x, y \mid x^p y^{-q} \rangle.$$

The abelianization is $x \mapsto t^q$ and $y \mapsto t^p$. We have

$$\begin{aligned}
\partial_1 x &= t^q - 1 \\
\partial_1 y &= t^p - 1 \\
\partial_2(x^p y^{-q}) &= (1 + t^q + t^{2q} + \cdots + t^{(p-1)q})x - (1 + t^p + t^{2p} + \cdots + t^{(p-1)q})y \\
&= \frac{1 - t^{pq}}{1 - t^q} x - \frac{1 - t^{pq}}{1 - t^p} y.
\end{aligned}$$

Since p and q are coprime, the polynomials $\frac{t^p-1}{t-1}$ and $\frac{t^q-1}{t-1}$ have no common factors (by the theory of cyclotomic polynomials), so

$$\ker \partial_1 = \mathbb{Z}[t^{\pm 1}]\left\langle \frac{t^p - 1}{t-1} x - \frac{t^q - 1}{t-1} y \right\rangle.$$

Therefore,

$$\Delta_{T_{p,q}}(t) = \frac{(1 - t^{pq})(t-1)}{(1-t^p)(1-t^q)}.$$

3.5. **The Wirtinger presentation.** Another presentation for a knot from its diagram (and better known) is the *Wirtinger presentation*. A basepoint is placed above the diagram, and generators are given by conjugating meridian loops by straight-line paths from the basepoint—only one generator per overstrand is needed. Relations are given by loops just below crossings. See, for example, [15, Exercise 1.2.22] for a more precise description.

The abelianization has the nice property that each generator, with the correct choice of orientations, is sent to t. Thus, $\partial_1 \mu_i = t - 1$ for each meridian generator μ_i, and the kernel of ∂_1 is generated by $\mu_2 - \mu_1, \dots, \mu_n - \mu_1$. By performing the substitutions $\mu_i' = \mu_i \mu_1^{-1}$ for $i \neq 1$, akin to the "canonical form" for the Dehn presentation, then the Alexander module is presented the matrix for ∂_2 with the column for μ_1 removed—the remaining columns are for $\mu_2', \dots, \mu_n'$). Remark 3.28 has the explicit calculation for ∂_2.

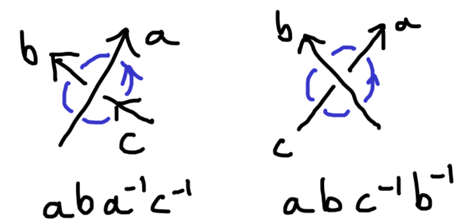
A property of the Wirtinger presentation of a knot with n crossings is that any one of the n crossing relations is a consequence of the $n-1$ others. Therefore, a square presentation matrix for the Alexander module can be obtained from taking a Wirtinger presentation of the knot and removing any row and column from the matrix for ∂_2. (This explains why the first elementary ideal tends to be used: $\mathcal{E}_{i+1}(\partial_2) = \mathcal{E}_i(H_1(\overline{S^3 - K}))$.)

Another property of the Wirtinger presentation is that, with it, it is fairly straightforward to see the following:

thm:partial-symmetry

Theorem 3.26. $\Delta_K(t) = \pm t^m \Delta_K(t^{-1})$ *for some* m

Proof. This is a simplification of [30]. Let mK denote the mirror image of K. Since there is a homeomorphism $S^3 - K \to S^3 - mK$, the groups $\pi_1(S^3 - K)$ and $\pi_1(S^3 - mK)$ are isomorphic. Take the Wirtinger presentation for $\pi_1(S^3 - K)$ as usual.



To think about mK, perform the reflection through the plane of the diagram, so in the resulting diagram what happens is that all the crossings switch types. If we reflect the basepoint through the diagram, too, then by reversing the direction of the nullhomotopic loop in the relations, a relation like $abc^{-1}b^{-1}$ transforms to $a^{-1}b^{-1}cb$. Thus, sending each generator to its inverse is a well-defined group homomorphism, and so $H_1(\overline{S^3 - mK})$ is isomorphic to $H_1(\overline{S^3 - K})$ as a $\mathbb{Z}[H_1(S^3 - K)]$-module. Therefore, $\Delta_K(t^{-1})$ is also the Alexander polynomial for K. □

Remark 3.27. This is a special fact about knot groups. For example, consider the HNN extension $G = \langle a, b, \mu \mid \mu a \mu^{-1} = b^2, \mu b \mu^{-1} = a \rangle$, whose abelianization is $H_1(G) = \langle t \rangle$.

$$\begin{aligned} \partial_1(a) &= \partial_1(b) = 0 \\ \partial_1(\mu) &= t - 1 \\ \partial_2(\mu a \mu^{-1} b^{-2}) &= ta - 2b \\ \partial_2(\mu b \mu^{-1} a^{-1}) &= tb - a \end{aligned}$$

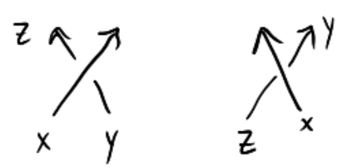
Therefore, $\ker \partial_1$ is generated by a and b and the presentation matrix for $H_1(\overline{X_G})$ is

$$\begin{bmatrix} t & -2 \\ -1 & t \end{bmatrix},$$

whose determinant is $\Delta_G(t) = t^2 - 2$. This is non-symmetric, so it cannot be the group of a knot complement.

remark:wirtinger-linear

Remark 3.28. We may as well be explicit here with the linearized Wirtinger relations. Consider these diagrams for right-handed and left-handed crossings:



The right-handed crossing gives the relation $xz = yx$, and the left-handed crossing gives $yx = xz$. With this labeling convention, both give through ∂_2 the relation

$$(1 - t)x - y + tz = 0.$$

Or, $z = (1 - t^{-1})x + t^{-1}y$, an affine linear combination. (This gives the structure of an Alexander quandle, where this is represented by the operation $z = x \triangleleft y$, for "y under x in the meridian direction." See Section 9.)

3.6. **Generalization to links.** All of the preceding discussion of the Alexander polynomial can be generalized to links. The group $H_1(S^3 - L)$ has rank equal to the number of link components by Alexander duality. If we choose a map $H_1(S^3-L) \to \mathbb{Z}$ corresponding to some orientations of each component, then we may use this in place of the abelianization to get $\Delta_L(t)$ for an oriented link.

Or, one can do everything for $\mathbb{Z}[H_1(S^3-L)]$-modules, where this ring is a multivariable Laurent polynomial ring, to get a multivariable Alexander polynomial.

Example 3.29. For a split unlink, the fundamental group is $\mathbb{Z} * \mathbb{Z}$ with abelianization $\mathbb{Z} \oplus \mathbb{Z}$. One can calculate $H_1(\overline{S^3 - L}) = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$, so $\Delta_L(s,t) = 0$.

Example 3.30. For the Hopf link, the fundamental group is $G = \mathbb{Z} \oplus \mathbb{Z}$, so $G'/G'' = 0$. Hence $\Delta_L(s,t) = 1$, the determinant of a 0×0 matrix.

3.7. **Fibered knots.** A knot or link L is called *fibered* if the complement $S^3 - \nu(L)$ fibers over S^1. (In other language: the knot or link is the *spine* of an open book decomposition of S^3, where the pages of the open book are the leaves of the fibering.)

The *monodromy* is a homeomorphism $g : \Sigma \to \Sigma$ for a compact surface with boundary such that $S^3 - \nu(L)$ is homeomorphic to the mapping torus

$$T_g = [0,1] \times \Sigma/(1,x) \sim (0, f(x)).$$

The fundamental group of a mapping torus is an HNN extension:

$$\pi_1(T_g) = \pi_1(T_g) *_{g_*} = \langle \pi_1(T_g), \mu \mid \mu x \mu^{-1} = g_*(x) \text{ for all } x \in \pi_1(T_g)\rangle.$$

Since the fundamental group of a surface with boundary is a free group, the abelianization of the HNN extension sends all of $\pi_1(T_g)$ to 1 and μ to t, hence the kernel of ∂_1 is generated by the generators of $\pi_1(T_g)$. We also have

$$\partial_2(\mu x \mu^{-1} g_*(x)^{-1}) = tx - g_*(x),$$

thus $H_1(\overline{S^3 - L})$ is presented by

$$H_1(\overline{S^3 - L}) = H_1(\Sigma; \mathbb{Z}[t^{\pm 1}])/(tx - g_*(x) \text{ for all } x).$$

Choosing a basis for $H_1(\Sigma)$ and letting A denote the matrix for $g_* : H_1(\Sigma) \to H_1(\Sigma)$, we see that $tI - A$ is a presentation matrix for the first Alexander module. Therefore,

Theorem 3.31. $\Delta_L(t) = \det(tI - A)$ *for a fibered link, where A is a matrix for the map $H_1(\Sigma) \to H_1(\Sigma)$ induced by the monodromy.*

Corollary 3.32. *The Alexander polynomial of a fibered link is monic.*

Thus, it is a necessary condition for K being fibered that $H_1(\overline{S^3 - K})$ is finitely generated as a $\mathbb{Z}$-module. Compare this to Theorem 3.19.

Since $\Delta_K(1) = \pm 1$ (Corollary 3.40), $H_1(\overline{S^3 - K})$ has no $\mathbb{Z}$-torsion, so, if $H_1(\overline{S^3 - K})$ is finitely generated as a $\mathbb{Z}$-module, it is a free $\mathbb{Z}$-module. Using Theorem 3.19, $\Delta_K(t)$ is monic iff $H_1(\overline{S^3 - K})$ is a finitely generated $\mathbb{Z}$-module.

Theorem 3.33 (Fibration theorem, [27]). *Given a compact irreducible 3-manifold M, a finitely generated group G not isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and a short exact sequence $1 \to G \to \pi_1(M) \to \mathbb{Z} \to 1$, then M fibers over S^1.*

Recall that $M = S^3 - \nu(K)$ is irreducible by the Sphere theorem, so if $G = [\pi_1(M), \pi_1(M)]$ is finitely generated, K is a fibered knot.

Example 3.34. The following facts are mostly drawn from [28] and KnotInfo [2].

- 5_2 is not fibered since its Alexander polynomial is $2t^2 - 3t + 2$.
- The trefoil and figure-eight knots *are* fibered and $\Delta_{3_1}(t) = 1 - t + t^2$ and $\Delta_{4_1}(t) = -1 + 3x - x^2$.
- For ≤ 10 crossings, having a monic Alexander polynomial is equivalent to the knot being fibered.
- The knot $11n_{73}$ has a nontrivial monic polynomial but it is not a fibered knot.
- For ≤ 11 crossings, having a monic Alexander polynomial whose breadth is twice the genus is equivalent to the knot being fibered.
- The knots $11n_{34}$ and $11n_{42}$ (respectively the Conway knot and the Kinoshita-Terasaka knot) both have trivial Alexander polynomial 1 and are not fibered.

Example 3.35. Torus knots are fibered. This can be shown through "Milnor fibers," but it also follows from Stallings's result. Using the presentation $G = \pi_1(S^3 - T_{p,q}) = \langle x, y \mid x^p = y^q \rangle$, we can examine $\overline{X_G}$ of the presentation complex X_G. The idea is that a closed loop in the 1-skeleton of $\overline{X_G}$ corresponds to elements of the commutator subgroup. The relator lifts to a loop that ascends through x's then descends through y's, and it has a certain "height." So, any loop in $\overline{X_G}$ whose "height" is at least as great as that of the relator can be reduced in height until its height is less. We only need to care about loops that do not visit the same 0-cell twice (since otherwise they are a product of two simpler loops). There are only finitely many such loops, essentially due to the fact that such loops come from primitive solutions of a system of integer linear equations.

sec:seifert-presentation

3.8. **Seifert presentation.** Recall that a Seifert surface Σ of an oriented link L is a compact oriented surface such that $\partial\Sigma = L$ and $\Sigma \cap L = \partial\Sigma$. Seifert surfaces exist for many reasons, for example because every link in S^3 is nullhomologous so there is a nontrivial class $H^2(S^3, L)$ whose boundary is L; it is a theorem that all second homology classes can be represented by embedded surfaces.

Proposition 3.36 ([20, Proposition 6.3])**.** *Suppose Σ is a connected Seifert surface. Then there is a a unique non-singular bilinear form*

$$\beta : H_1(S^3 - \Sigma) \times H_1(\Sigma) \to \mathbb{Z}$$

with the property that $\beta([c],[d]) = \mathrm{lk}(c,d)$ for any homology classes represented by oriented simple closed curves c and d.

Proof. This is Alexander duality. □

Definition 3.37. The *Seifert form* of a Seifert surface Σ is a bilinear form

$$\alpha : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$$

defined by $\alpha([c],[d]) = \mathrm{lk}(c, i^+ d)$ for any homology classes represented by oriented simple closed curves c and d, where $i^\pm : \Sigma \to \Sigma \times \{\pm 1\}$ are the pushoffs using a homeomorphism $\nu(\Sigma) \cong \Sigma \times [-1,1]$ for the embedded normal bundle.

thm:seifert-square-pres

Theorem 3.38 ([20, Theorem 6.5]). *Let A be a matrix for the Seifert form α of a connected Seifert surface Σ of an oriented link L. Then $tA - A^T$ is a presentation matrix for $H_1(\overline{S^3 - L})$. In particular, $\Delta_L(t) = \det(tA - A^T)$.*

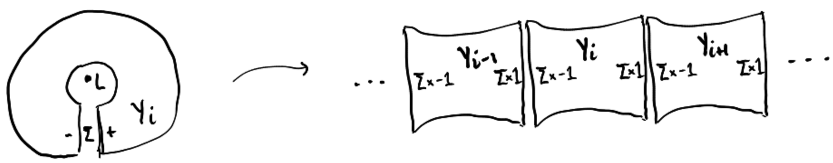
Proof. Choose a basis $f_1, \dots, f_n$ of simple closed curves for $H_1(\Sigma)$, and let $e_1, \dots, e_n$ be a corresponding dual basis of simple closed curves for $H_1(S^3 - \Sigma)$. That is, $\beta(e_i, f_j) = \delta_{ij}$. Notice $A_{ij} = \alpha(f_i, f_j) = \operatorname{lk}(f_i, i^+ f_j) = \beta(i^+ f_j, f_i)$ and $A_{ij} = \operatorname{lk}(f_i, i^+ f_j) = \operatorname{lk}(i^- f_i, f_j) = \beta(i^- f_j, f_i)$ mean

$$i_*^+[f_i] = \sum_j A_{ij}[e_j] \quad \text{and} \quad i_*^-[f_j] = \sum_i A_{ij}[e_i].$$

One can construct $\overline{S^3 - \Sigma}$ by

- letting, for each $i \in \mathbb{Z}$, Y_i denote a copy of $S^3 - \nu(\Sigma) - \nu(L)$, then
- gluing the $\Sigma \times \{+1\}$ side of Y_{i-1} to the $\Sigma \times \{-1\}$ side of Y_i for all i.

This has $\langle t \rangle$ as the group of deck transformations by having t send a point of Y_i to the corresponding point of Y_{i+1}.



Just like in Mayer–Vietoris, we can cut up chains so that they are sums of chains from each Y_i, and applying the $\mathbb{Z}[t^{\pm 1}]$ action we can represent every chain as being an element of $\mathbb{Z}[t^{\pm 1}] \otimes C_i(Y_0)$, so every cycle can be written as a $\mathbb{Z}[t^{\pm 1}]$-linear combination of cycles from $C_i(Y_0)$. Since the loops $e_1, \dots, e_n$ do not intersect Σ, they lift to loops in $\overline{S^3 - \Sigma}$, so then every 1-cycle is homologous to a $\mathbb{Z}[t^{\pm 1}]$-linear combination of these loops. By duality, we can also use $f_1, \dots, f_n$ (pushed off by i^+) in Y_0 as a generating set for 1-cycles. Since $t[f_j] = t[i^- f_j] = \sum_i A_{ij}[e_i]$ and $[f_i] = [i^+ f_i] = \sum_j A_{ij}[e_j]$, we can deduce $tA_{ij}[e_j] = A_{ji}[e_i]$, so $tA - A^T$ is a matrix of relators for $[e_1], \dots, [e_n]$. With some more effort, one can check that $tA - A^T$ is indeed a presentation matrix for $H_1(\overline{S^3 - \Sigma})$, for example by appealing to Mayer–Vietoris with $Y' = \bigcup_{\text{even } i} Y_i$ and $Y'' = \bigcup_{\text{odd } i} Y_i$. □

Corollary 3.39. *For a link L, the genus satisfies $2g(L) + c - 1 \geq \operatorname{breadth} \Delta_L(t)$, where c is the number of components in L and the breadth is the difference between the maximal and minimal degrees of t.*

cor:poly-at-1

Corollary 3.40. *For K a knot, $\Delta_K(1) = \pm 1$.*

Proof. Let Σ be a connected Seifert surface for K, which is a once-punctured genus-g surface. Choose a symplectic basis for $H_1(\Sigma)$, which is a basis $f_1, f_2, \dots, f_{2g-1}, f_{2g}$ such that the algebraic intersection number has a matrix that is block diagonal, with each block being the 2×2 matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We have $\Delta_K(1) = \pm\det(A - A^T)$. Notice $(A - A^T)_{ij} = \mathrm{lk}(f_i, i^+ f_j) - \mathrm{lk}(f_j, i^+ f_i) = \mathrm{lk}(i^- f_i, f_j) - \mathrm{lk}(i^+ f_i, f_j)$, which is the algebraic intersection number of f_i and f_j. Thus $\det(A - A^T) = \left(\det\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)^g = 1$. □

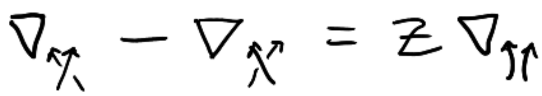
Corollary 3.41. *For K a knot, there is an Alexander polynomial representative such that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$.*

3.9. **Duality.**

4. The Conway potential

sec:conway-potential

In [4], Conway defined the *potential function*, now known as the *Alexander-Conway polynomial*, a polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ associated to oriented links L. It is completely determined by $\nabla_{\mathrm{unknot}}(z) = 1$ and the *skein relation*



where the three links are the same outside the sphere that contains the portrayed tangles. (Note that this is an invariant of links, not of link diagrams!)

That this is well-defined takes some proof. One way to proceed is to use the fact that every link can be unknotted by switching crossings bounded by the crossing number of the link and inducting on crossing number. We will instead follow Conway's lead and connect it to a normalized version of the Alexander polynomial.

Conway reported that, by using his rational-tangle-based knot notation and this potential function, he was able to undertake the enumeration of all 54 knots up to 11 crossings in an afternoon, rather than Little's six years.

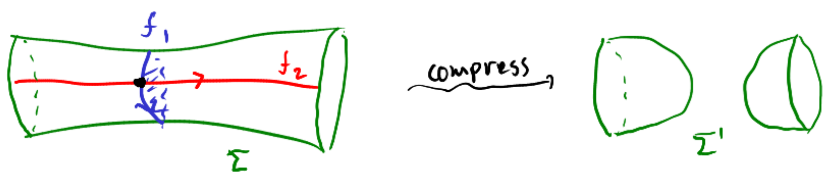
Remark 4.1. It was known to Alexander that there was a way to get a three-term relation with some care [1, (12.2)]. The beauty of the Conway potential is that it is normalized so linear relations automatically make sense. The substitution $\nabla_L(t^{1/2} - t^{-1/2})$, as the following theorem shows, gives the symmetric representative of the Alexander polynomial of a knot. (That is, for a knot the substitution is in $\mathbb{Z}[t^{\pm 1}]$!) A modern formulation of Alexander's approach will be described in Section 4.1.

Remark 4.2. The original Conway potential is multivariate, with each component of the link being given its own variable; the single-variable case is when each component is given the same variable. The multivariable version is defined by symmetry properties, a component deletion formula, and a connect sum formula [4, Section 6]. He gives the above skein relation (between two strings with the same variable) as a consequence, along with another skein relation between two strings with different variables. He writes that "we have not found a satisfactory *explanation* of these identities, although we have *verified* them by reference to a 'normalized' form of the 'L-matrix' definition of the Alexander polynomial, obtained by associating the rows and columns in a natural way. [...] It seems plain that much work remains to be done in this field." This 'L-matrix' definition is given in the following theorem, at least for the single-variable case.

Theorem 4.3. *Let A be the matrix of the Seifert form for a connected Seifert surface for an oriented link L. Then* $\nabla_L(t^{1/2} - t^{-1/2}) = \det(t^{1/2}A - t^{-1/2}A^T)$.

Proof. Every smooth connected Seifert surface can be represented as the regular fiber of a map $S^3 - \nu(L) \to S^1$. Given two Seifert surfaces, by taking a stable representative of a family $I \times (S^3 - \nu(L)) \to S^1$, one can use Cerf theory to see that all Seifert surfaces are related by a sequence of compressions ("embedded 1-surgeries") and "de-compressions" ("embedded 0-surgeries"). With a little care, we can make sure that the Seifert surfaces remain connected throughout these operations. So, all we need to do is check that compressing a Seifert surface leaves $\det(t^{1/2}A - t^{-1/2}A^T)$ invariant.

Let D be a compression disk for a connected Seifert surface Σ, where the compressed surface Σ' is connected. Choose a basis for Σ by taking ∂D as one curve f_1 then extending it to a symplectic basis, with f_2 being the curve that intersects f_1 with algebraic intersection number ± 1.



Then with the appropriate orientations, the Seifert matrix A_Σ of Σ can be written in terms of the Seifert matrix $A_{\Sigma'}$ of Σ':

$$A_\Sigma = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 1 & c_2 & c_3 & \cdots & c_n \\ \hline 0 & * & & & \\ \vdots & \vdots & & A_{\Sigma'} & \\ 0 & * & & & \end{array}\right].$$

With

$$P = \begin{bmatrix} 1 & -c_2 & \cdots & -c_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

we have

$$PA_\Sigma P^T = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & * & & & \\ \vdots & \vdots & & A_{\Sigma'} & \\ 0 & * & & & \end{array}\right].$$

This P is a basis change for a bilinear form, and replacing A_Σ by $PA_\Sigma P^T$ in the determinant just multiplies the determinant by $\det(P)^2 = 1$, so we can assume the A_Σ is in this new form. Then, since

$$t^{1/2}A_\Sigma - t^{-1/2}A_\Sigma^T = \left[\begin{array}{cc|ccc} 0 & -t^{-1/2} & 0 & \cdots & 0 \\ t^{1/2} & 0 & * & \cdots & * \\ \hline 0 & * & & & \\ \vdots & \vdots & & t^{1/2}A_{\Sigma'} - t^{-1/2}A_{\Sigma'}^T & \\ 0 & * & & & \end{array}\right],$$

when we expand the determinant along the first row then the first column, this reduces to $\det(t^{1/2}A_{\Sigma'} - t^{-1/2}A_{\Sigma'}^T)$. Therefore, the determinant is an invariant of the link. □

Corollary 4.4. *If L is a split link, then $\nabla_L(z) = 0$. More generally, if L bounds a disconnected minimal-genus Seifert surface with no sphere components, then $\nabla_L(z) = 0$.*

Proof. There is a connected Seifert surface for L which has a separating simple closed curve. This means $t^{1/2}A - t^{-1/2}A^T$ has a row and column of all 0's if we take that curve as part of the basis for H_1. □

Corollary 4.5. *If $K = K_1 \# K_2$, then $\nabla_K(z) = \nabla_{K_1}(z)\nabla_{K_2}(z)$.*

Proof. Given Seifert surfaces Σ_1 and Σ_2 for K_1 and K_2, respectively, one gets a Seifert surface Σ for K by joining them along an arc in the boundary. A basis for $H_1(\Sigma) \cong H_1(\Sigma_1) \oplus H_1(\Sigma_2)$ is given by the concatenation of bases for each. Thus, the Seifert matrix for K is

$$A_K = \begin{bmatrix} A_{K_1} & \\ & A_{K_2} \end{bmatrix}.$$

The result follows. □

sec:alexander-three-term

4.1. **Alexander's three-term relation.** In this section, we go through a generalization of Alexander's observation from [1, (12.2)]. We will take an operad or planar algebra approach to Alexander modules for certain manifolds with corners.[1] It should be said that all of this will be a long-winded way to say that we can make square presentation matrices for the Alexander modules for the three knots appearing in the Conway skein relation so that they are the same but for their first rows, and since the sum of their first rows will be 0, therefore their determinants sum to zero by multilinearity.

First, let us revise the construction of the first Alexander module to use homology with local coefficients (see [15, Section 3.H]). Let M be a compact manifold with a homomorphism $f : \pi_1(M) \to \langle t \rangle$. The group ring $\mathbb{Z}[\langle t \rangle] \cong \mathbb{Z}[t^{\pm 1}]$ is a $(\mathbb{Z}[\pi_1(M)], \mathbb{Z}[\pi_1(M)])$-bimodule, where the action is given by multiplication by the image through f. Define a chain complex $C_n(M; \mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[\pi_1(M)]} C_n(\widetilde{M})$ with boundary maps $\partial \otimes \mathrm{id}$, where $\widetilde{M} \to M$ is the universal cover. The homology $H_n(M; \mathbb{Z}[t^{\pm 1}])$ of this complex is a case of *homology with local coefficients*. (In general, any right $\mathbb{Z}[\pi_1(M)]$-module may be used as the local coefficients.) Due to the bimodule structure, the homology groups are themselves $\mathbb{Z}[t^{\pm 1}]$-modules.

Notice that for $p(t) \otimes \sigma \in C_n(M; \mathbb{Z}[t^{\pm 1}])$, if $g \in \ker f$ then $p(t) \otimes g\sigma = p(t) \otimes \sigma$. Thus, if $\overline{M} \to M$ is the infinite cyclic cover associated to f (where if f is not surjective then $\overline{M}$ is a disjoint union of a copy of M for each element of $\langle t \rangle$), then $C_n(M; \mathbb{Z}[t^{\pm 1}]) \cong C_n(\overline{M})$. Hence,

[1] I wonder if one might potentially be able to cast something like this as a "topological field theory."

in the case that f is the abelianization map, $H_n(M;\mathbb{Z}[t^{\pm 1}])$ is isomorphic to $H_n(\overline{M};\mathbb{Z})$ as $\mathbb{Z}[t^{\pm 1}]$-modules.

Definition 4.6. For $k \in \mathbb{Z}_{\geq 0}$, let S^3_k denote the *k-times punctured* 3*-sphere*, a compact oriented 3-manifold that is homeomorphic to the result of removing from the 3-sphere k open balls whose closures are disjoint.

Definition 4.7. A *tangle* is a properly embedded[2] compact oriented 1-manifold T in some S^3_k such that each component of ∂S^3_k has an equal number of elements of ∂T with positive and negative orientations.

Tangle equivalence is isotopy of T rel ∂T. The constraint on boundary orientations exactly characterizes tangles as being the result of taking an oriented link L in S^3 then removing some number of balls whose boundaries intersect L transversely. Tangles have planar diagrams just like links, but the diagrams are given on punctured S^2, with each boundary component of the diagram corresponding to a particular boundary component of S^3_k. When working with planar algebras, it is customary to put one of the punctures at infinity. For example, here is a tangle in S^3_2:

To be perfectly clear with these diagrams, we ought to (1) label each boundary component and (2) establish a convention on the identities of each boundary point of ∂T (for example, mark one boundary point as "first" then require the orientations to alternate +/− as one goes around a boundary circle).

There is a composition law for compatible tangles, where two tangles are glued along a chosen pair of boundary components in a way that respects all orientations and connects up their respective 1-manifolds. What we are going for is that the skein relation for the Conway polynomial can be put in terms of compositions of pairs of S^3_1 tangles with four boundary points each.

Let (S^3_k, T) be a tangle. The same sort of argument that gives the Wirtinger presentation of a link complement also works for $\pi_1(S^3_k - T)$, giving a group presentation with meridian loops as generators, but in addition to relations from crossings there is one additional relation per boundary component—these are loops that parallel boundary components in a diagram, sitting just underneath each strand of T. Then there is a homomorphism $f : \pi_1(S^3_k, T) \to \langle t \rangle$ sending each meridian to t using orientation of the tangle. Thus, we have an *Alexander module* $H_1(S^3_k - T; \mathbb{Z}[t^{\pm 1}])$ for the tangle. [For another approach, we can calculate $H^1(S^3_k - T)$ using Alexander duality by recognizing that $S^3_k - T = S^3 - (T \cup \bigcup_{i=1}^k B_i)$, where the B_i the balls for which $S^3_k = S^3 - \bigcup_{i=1}^k B_i$. The space $T \cup \bigcup_{i=1}^k B_i$ is homotopy equivalent to a graph with oriented edges. As a trick, fill in each boundary component of the tangle in some way to get a link (S^3, L). There is

[2] In 3-manifold topology, T is properly embedded in M iff T is embedded in M and $\partial M \cap T = \partial T$ is a transverse intersection.

an element of $H^1(S^3 - L)$ that calculates the linking number with L (as, say, the Poincaré dual of a Seifert surface for L). The restriction of this element to $\alpha \in H^1(S^3_k, T)$ is the one that calculates linking number with T, so there is a homomorphism $f : \pi_1(S^3_k - T) \to \langle t \rangle$.]

Consider a boundary component $\Sigma \subset \partial S^3_k$. There is an Alexander module $H_1(\Sigma - \partial T; \mathbb{Z}[t^{\pm 1}])$ of this boundary, too, with the homomorphism $\pi_1(\Sigma - \partial T) \to \langle t \rangle$ from composition with the induced map $\pi_1(\Sigma - \partial T \hookrightarrow S^3_k - T)$.[3] To take into account all the peripheral Alexander modules, we give the following definition:

Definition 4.8. The *precise Alexander module* of a tangle (S^3_k, T) is the following data: the module $H_1(S^3_k - T; \mathbb{Z}[t^{\pm 1}])$, the peripheral Alexander modules $H_1(\Sigma_i - \partial T; \mathbb{Z}[t^{\pm 1}])$ for $\Sigma_1, \dots, \Sigma_k$ the boundary spheres of S^3_k, and the map

$$\bigoplus_{i=1}^{k} H_1(\Sigma_i - \partial T; \mathbb{Z}[t^{\pm 1}]) \to H_1(S^3_k - T; \mathbb{Z}[t^{\pm 1}])$$

induced by the inclusions $\Sigma_i - \partial T \hookrightarrow S^3_k - T$. (We have not been too careful about basepoints: technically, we must choose a basepoint for each boundary component of S^3_k and a path from each to the basepoint of S^3_k itself, all of in the complement of T.)

We will now work out how to compute the precise Alexander module of the composition of two tangles. Let (S^3_k, T) and $(S^3_{k'}, T')$ be two tangles that can be composed along some boundary components $\Sigma \subset \partial S^3_k$ and $\Sigma' \subset \partial S^3_{k'}$, and let $(S^3_{k+k'-2}, T \cup T')$ denote the composition. That is, there is some orientation-reversing homeomorphism $h : \Sigma \to \Sigma'$ carrying each boundary point of $\Sigma \cap \partial T$ to a corresponding point of $\Sigma' \cap \partial T'$, again with reversed orientation. Suppose also that $\Sigma \cap \partial T$ is nonempty. The claim is that the Alexander module of the composition is the fibered coproduct[4]

$$H_1(S^3_{k+k'-2} - T \cup T'; \mathbb{Z}[t^{\pm 1}]) \cong H_1(S^3_k - T; \mathbb{Z}[t^{\pm 1}]) \oplus_{H_1(\Sigma - \partial T; \mathbb{Z}[t^{\pm 1}])} H_1(S^3_{k'} - T'; \mathbb{Z}[t^{\pm 1}]).$$

Consider each of these Alexander modules as being the homology of an infinite cyclic cover. Then the homology of a composition of tangles can be computed using the reduced Mayer–Vietoris sequence:

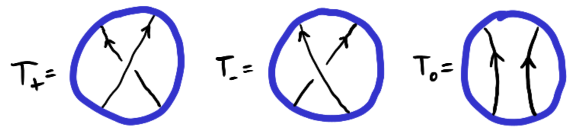
$$H_1(\overline{\Sigma}) \to H_1(\overline{S^3_k - T}) \oplus H_1(\overline{S^3_{k'} - T'}) \to H_1(\overline{S^3_{k+k'-2} - T \cup T'}) \to \widetilde{H}_0(\overline{\Sigma}).$$

Since we assumed $\Sigma \cap \partial T$ was nonempty, $\overline{\Sigma}$ is connected so $\widetilde{H}_0(\overline{\Sigma}) = 0$. Thus, the Alexander module of the composition is the fibered coproduct as claimed. The precise Alexander module of the composition is from taking this fibered coproduct along with the set of Alexander modules for each of the remaining boundary components of both tangles.

Let us compute the precise Alexander modules of the following three tangles in S^3_1:

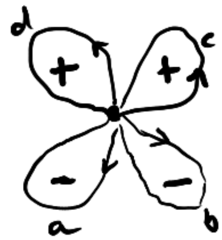
[3] If we identify Σ with the Riemann sphere and let $c_1, \dots, c_n$ be the points of $\partial T \cap \Sigma$ with respective orientations $\epsilon_1, \dots, \epsilon_n$, then applying the argument principle to $\prod_{i=1}^{n} (z - c_i)^{\epsilon_i}$ on $\Sigma - \partial T$ gives the homomorphism to $\mathbb{Z}$.

[4] Recall: a *fibered coproduct* or *pushout* $A \oplus_B C$ with two maps $f : B \to A$ and $g : B \to C$ is the quotient $(A \oplus C)/\{(f(b), 0) - (0, g(b))$ for all $b \in B\}$.



These are all examples of *rational tangles* (defined by Conway in [4]), which are tangles (S^3_1, T) characterized by $S^3_1 - T$ being homotopy equivalent to a space that is the union of $S^2 - \{\text{four points}\}$ and a disk whose boundary separates the four points into two groups of two points; or, equivalently, that (S^3_1, T) is isotopic to (S^3_1, T_0) while allowing the boundary points to move freely. While the Alexander modules for these are all the same, the maps $H_1(\Sigma - \partial T; \mathbb{Z}[t^{\pm 1}]) \to H_1(S^3_1 - T; \mathbb{Z}[t^{\pm 1}])$ are all different.

The boundary $\Sigma - \partial T$ is a four-times punctured 2-sphere, so $\pi_1(\Sigma - \partial T)$ is a free group on three generators. Concretely, with the generators as in the following diagram, we have a presentation $\langle a, b, c, d \mid a^{-1}b^{-1}cd = 1\rangle$ for $\pi_1(\Sigma - \partial T)$, and the homomorphism $\pi_1(\Sigma - \partial T) \to \langle t \rangle$ is given by $a, b, c, d \mapsto t$.



(A way to construct $\overline{\Sigma - \partial T}$ is to slice $\Sigma - \partial T$ open along two disjoint arcs that connect a $+$ to a $-$, take a $\mathbb{Z}$'s worth of them in a stack, then glue sides of arcs together so that a path going counterclockwise around a $+$ point goes from level n to level $n+1$, just like in the construction of the infinite cyclic cover using a Seifert surface, but down a dimension.)

We have that ∂_1 of each generator is $t - 1$ and that

$$\partial_2(a^{-1}b^{-1}cd) = a^{-1} + t^{-1}b^{-1} + t^{-2}c + t^{-1}d = t^{-2}(-ta - b + c + td).$$

Let $K \subset \mathbb{Z}[t^{\pm 1}]\langle a, b, c, d\rangle$ denote the kernel of ∂_1, which is those elements $x_1 a + x_2 b + x_3 c + x_4 d$ such that $x_1 + x_2 + x_3 + x_4 = 0$. Then $H_1(\Sigma - \partial T; \mathbb{Z}[t^{\pm 1}]) \cong K/(-ta - b + c + td)$. [Letting $a' = a$, $b' = b - a$, $c' = c - a$, and $d' = d - a$, the relation is $td' + c' - b'$ and K is generated by b', c', d', which implies $H_1(\Sigma - \partial T; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]\langle b', c', d'\rangle/(td' + c' - b') \cong \mathbb{Z}[t^{\pm 1}]^2$.]

Gluing a disk into $\Sigma - \partial T$ whose boundary separates the four punctures into two groups of two has the effect of adding an additional relation to the Alexander module. For example, with T_0 the disk is glued along the path $b^{-1}c$. Since $\partial_2(b^{-1}c) = -t^{-1}b + t^{-1}c$,

$$H_1(S^3_1 - T_0; \mathbb{Z}[t^{\pm 1}]) \cong K/(t(d-a) + c - b, c - b).$$

For T_+, the disk is glued along ca^{-1}. Since $\partial_2(ca^{-1}) = c - a$,

$$H_1(S^3_1 - T_+; \mathbb{Z}[t^{\pm 1}]) \cong K/(t(d-a) + c - b, c - a).$$

Lastly, for T_-, the disk is glued along db^{-1}. Since $\partial_2(db^{-1}) = d - b$,

$$H_1(S^3_1 - T_-; \mathbb{Z}[t^{\pm 1}]) \cong K/(t(d-a) + c - b, d - b).$$

At this point, notice that $t(c-a) + t(d-b) + (1-t)(c-b) = t(d-a) + c - b$. This underpins Alexander's three-term relation.

Given a diagram of a link L with n crossings, by removing every crossing we get a tangle T in S^3_n, and $S^3_n - T$ is homotopy equivalent to the complement of a 4-regular graph in S^3. The Wirtinger presentation of this tangle has one meridian generator per edge of this graph, and each removed crossing gives a relation like $a^{-1}b^{-1}cd = 1$, like our recent calculation of $\pi_1(S^2 - \text{four points})$. Like usual for the Wirtinger presentation, any one of these relations is a consequence of the other $n-1$ relations. Reversing this: we may start with a planar 4-regular graph with edges oriented so that every vertex has an equal number of incoming and outgoing edges—this is a tangle (S^3_n, T). Then for each vertex we may choose a rational tangle to insert there, giving a link L.

The Alexander module for L is the fibered coproduct of the Alexander module of (S^3_n, T) along with all the Alexander modules for each of the vertex tangles. Since the relations for the Alexander module for (S^3_n, T) are already distributed among the modules for each of the vertices, all $H_1(S^3_n - T; \mathbb{Z}[t^{\pm 1}])$ does in the fibered coproduct is to "join up" corresponding meridians for each module. Thus, the Alexander module for $S^3 - L$ can be described as follows. Let $a_1, \dots, a_{2n}$ denote the meridian generators for $S^3_n - T$, and let K be the kernel of the map $\mathbb{Z}[t^{\pm 1}]\langle a_1, \dots, a_{2n}\rangle \mapsto \mathbb{Z}$ where $a_i \mapsto 1$. Then a matrix for ∂_2 can be given by the n rows corresponding to the $t(d-a)+c-b$ elements followed by n rows for the specific relation for each vertex, which for T_0, T_+, and T_- are $c-b$, $c-a$, and $d-b$, respectively. Thus, ∂_2 has an $2n \times 2n$ matrix, though any one of the first n rows may be removed.

A generating set for K can be obtained by taking $a_i' = a_i - a_1$ for all $2 \leq i \leq 2n$. Then, for example, $t(d-a)+c-b = t(d'-a')+c'-b'$. After removing one of the first n rows then rewriting in terms of this basis, ∂_2 is a $(2n-1)\times(2n-1)$ matrix.

Each choice of T_+, T_-, or T_0 at a particular crossing gives three matrices with the corresponding row corresponding to that tangle's relation. Since $t(c-a)+t(d-b)+(1-t)(c-b) = t(d-a)+c-b$, by multilinearity of determinants, we have with this choice of presentation matrices that

$$t\Delta_{L_+}(t) + t\Delta_{L_-}(t) + (1-t)\Delta_{L_0}(t) = 0,$$

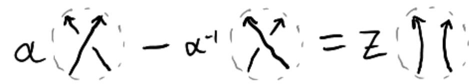
since $t(d-a)+c-b$ after the basis change is a linear combination of the other rows in the matrix.

Remark 4.9. It would be nice to have a formula for the Alexander module of a general rational tangle. There also seems to be a good amount to say about using the decomposition from bridge position into two tangles. I would hope for something about the exact structure of the Alexander modules (like, what, geometrically speaking, causes the higher elementary ideals to be nontrivial?).

5. The HOMFLY-PT polynomial

Immediately after Jones introduced his 1-variable polynomial invariant of links in [16], five groups of mathematicians (Freyd–Yetter, Lickorish–Millet, Ocneanu, Hoste, and Przytycki–Traczyk) independently discovered a 2-variable polynomial invariant that specialized to both the Alexander polynomial and the Jones polynomial. The first four groups combined their results into a single paper [13], from whom the first letters of each of their last names were borrowed to name the HOMFLY polynomial. Due to slow postal service, [23] did not arrive in time to be part of the paper, so it is also known as the HOMFLY-PT polynomial to recognize the work of Przytycki and Traczyk.

From a skein-module-like point of view, the HOMFLY-PT polynomial arises from the following construction. Let $\mathscr{L}$ be the $\mathbb{Z}[\alpha^{\pm}, z]$-module freely generated by isotopy classes of nonempty oriented links in S^3. Let $\mathscr{A}$ be the quotient of $\mathscr{L}$ by the skein relation



where the three links are the same outside the dotted circles (each dotted circle represents an S^2 that intersects the link in exactly four points, where one side of that sphere is one of the three rational tangles pictured). One can show that $\mathscr{A} \cong \mathbb{Z}[\alpha^{\pm}, z]\cdot$unknot by induction on the number of crossings in the diagram (see [20, Chapter 15] for a complete proof). The HOMFLY-PT polynomial $P_L(\alpha, z)$ is from setting the value of the unknot to 1 (or, from taking the value of L in the quotient $\mathscr{A}/(\text{unknot} - 1) \cong \mathbb{Z}[\alpha^{\pm}, z]$).

Remark 5.1. The substitution $\alpha = 1$ gives the Conway potential from Section 4.

Remark 5.2. There are *many* parameterizations of the HOMFLY-PT polynomial. Essentially, the HOMFLY-PT polynomial is a homogeneous 3-variable polynomial which can be deprojectivized in any number of ways to yield a 2-variable polynomial. For example, if L_+, L_-, and L_0 are the three links from the skein relation, any of the following skein relations give a version of the HOMFLY-PT polynomial that appears in the literature:

- $\alpha L_+ - \alpha^{-1} L_- = z L_0$ (from above, for completeness)
- $\ell L_+ + \ell^{-1} L_- + m L_0 = 0$
- $x L_+ + y L_- + z L_0 = 0$

The original approach to the Jones polynomial was to take a braid whose closure was a given link, feed it through a new braid representation from subfactors of von Neumann algebras, then take a trace of the resulting linear operator. Turaev in [31] extended this to representations of affine Lie algebras of various types, and one particular case is link invariants from the fundamental representation of the quantum group $\mathcal{U}_q(\mathfrak{sl}(N))$. Interestingly, the resulting link invariant is characterized by the skein relation

$$q^N L_+ - q^{-N} L_- = (q - q^{-1}) L_0,$$

which with the substitution $\alpha = q^N$ and $z = q - q^{-1}$ is the HOMFLY-PT polynomial. That is, the HOMFLY-PT is an interpolation of the $\mathcal{U}_q(\mathfrak{sl}(N))$ invariants for all $N \in \mathbb{Z}_{\geq 1}$!

The Alexander polynomial then is, mysteriously, the quantum invariant associated to "$\mathcal{U}_q(\mathfrak{sl}(0))$." (I have been told that it is possible that this might be identified with the invariant for $\mathcal{U}_q(\mathfrak{gl}(1|1))$, which is the quantum deformation of the universal enveloping algebra for the superalgebra $\mathfrak{gl}(1|1)$. See [25] for a description of the Alexander polynomial as a $\mathcal{U}_q(\mathfrak{gl}(1|1))$ quantum invariant.)

Remark 5.3. The Jones polynomial is the $\mathcal{U}_q(\mathfrak{sl}(2))$ invariant from the fundamental representation, and in the usual parameterization it corresponds to the substitution $\alpha = t^{-1}$ and $z = t^{1/2} - t^{-1/2}$ in the HOMFLY-PT polynomial.

Remark 5.4. At $N = 2$ and $q = i$, the skein relation becomes

$$L_+ - L_- = -2i L_0.$$

This is the Conway potential at $z = -2i$ (and, incidentally, an evaluation of the Jones polynomial). Murakami showed that the general Alexander polynomial comes from quantum invariants of other representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ at $q = i$. See [32] and its references.

6. Kauffman state sum

By carefully considering Alexander's original definition, Kauffman devised a state sum model for the Alexander polynomial in *Formal Knot Theory* [19] (with a 2006 supplementary paper [18]).[5] Links are treated completely combinatorially as diagrams. The shadow of a diagram is a four-regular planar graph that he calls a *link universe*. The data of a link universe is the underlying abstract graph along with the *rotation system* at each vertex, which is the counterclockwise order of incident edges at each vertex. Elsewhere in mathematics, link universes are known as four-regular planar combinatorial maps.

Recall the construction of the associated matrix from the Dehn presentation, whose rows come from the rule illustrated in Figure 3. The original version of the associated matrix was an $n \times (n+2)$ matrix with one column per region, and in our exposition we removed the column associated to the "outer" region. To calculate the Alexander polynomial, Definition 2.6 says to also remove a column associated to a region neighboring the "outer" region, so now we have an $n \times n$ matrix A—call both the "outer" region and the chosen neighboring region the *starred* regions. Consider the expansion

$$\det A = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

Each nonzero term in this summation corresponds to a choice at each crossing of an unstarred region such that no region is chosen by two different crossings. Kauffman indicates these *states* on a link universe as a collection of *markers* between two incident edges at a vertex. For example, Figure 4 shows the three states of a trefoil universe (with the starred regions indicated by asterisks), and the markers are indicated by quarter circles at crossings. The rule from Figure 3 is used to label corners of regions in the link universe as in Figure 5.

Figure 4. The three states for a trefoil universe. fig:trefoil-universe-states

Hence, the Alexander polynomial is the sum over all the states of the products of the weights at each of its markers, times the sign of the permutation corresponding to that state. Up to a global value $\epsilon = \pm 1$, for a state S associated to a permutation $\sigma \in S^n$,

[5] In my "just so" story, his work with state sums put him in a unique position to spring into action when the Jones polynomial would appear two years later, where he immediately came up with the Kauffman bracket formulation. However, I am told that the Kauffman bracket was something he came up with for another purpose, and only later did he recognize it computed the Jones polynomial.

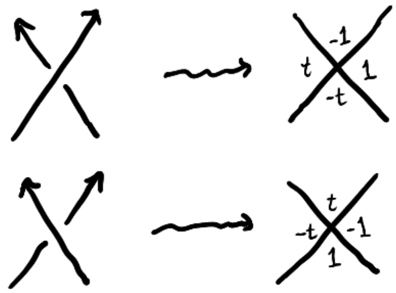


FIGURE 5. Alexander weights for a link universe. fig:alexander-universe

FIGURE 6. A black hole. fig:black-hole

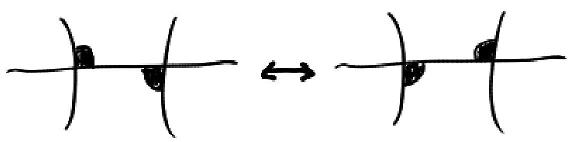


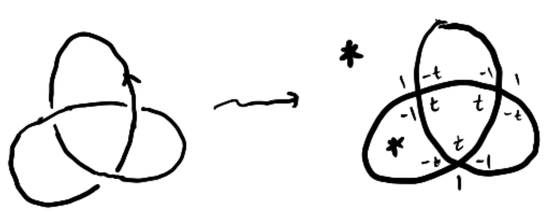
FIGURE 7. The Clock Theorem says every pair of states is connected by a sequence of these "clock moves." fig:clock-move

the sign $(-1)^\sigma$ is $\epsilon(-1)^{b(S)}$, where $b(S)$ counts the number of markers that are *black holes* (Figure 6)—this follows from [19, The Clock Theorem] (pictured in Figure 7). Therefore,

$$\Delta_L(t) = \sum_S (-1)^{b(S)} \langle L \mid S \rangle,$$

where the sum ranges over all states of a marked link universe for L, and $\langle L \mid S \rangle$ is the product of all the Alexander weights at the markers for S.

Example 6.1. The trefoil knot has the following labeled link universe:



The three states in Figure 4 give values of $\langle K \mid S \rangle$ of t, t^2, and t^3, respectively. The numbers of black holes in each state are respectively 0, 1, and 2. Therefore the Alexander polynomial of the trefoil is $t - t^2 + t^3 = t(1 - t + t^2)$.

By the Clock Theorem, one can show that using the weights $t, t, 1, 1$ instead of $t, -t, 1, -1$ gives the same polynomial, up to an overall multiplication by ± 1. Dividing the weights by $t^{1/2}$ gives $t^{1/2}, t^{1/2}, t^{-1/2}, t^{-1/2}$, which changes the polynomial only by a factor of some power of $t^{1/2}$. Again by the Clock Theorem, weights at locations that are between oppositely oriented arcs can be set to 1, changing the polynomial by a factor of some power of $t^{1/2}$. The resulting weights are illustrated in Figure 8.

Theorem 6.2. *The state sum for the weights in Figure 8 is a polynomial in $\mathbb{Z}[t^{\pm 1}]$ that is invariant under the Reidemeister moves, and it coincides with the Conway potential (Section 4) with the substitution $z = t^{-1/2} - t^{1/2}$.*

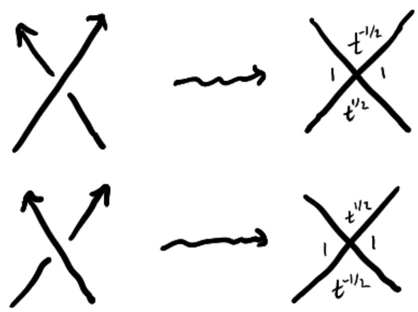


FIGURE 8. Renormalized Alexander weights. fig:alexander-weights-renormalized

Example 6.3. The *knot determinant* is the value $|\Delta_K(-1)|$. It governs whether nontrivial Fox n-colorings exist for a knot, which group theoretically are homomorphims $\pi_1(S^3 - K) \to D_n$ to the dihedral group such that meridians are sent to flips of the n-gon. The flips in D_n can be associated to elements of $\mathbb{Z}/n\mathbb{Z}$, with τ_k for $k \in \mathbb{Z}/n\mathbb{Z}$ being a flip composed with a rotation by $2\pi k/n$. Flips τ_i and τ_j indexed by $i, j \in \mathbb{Z}/n\mathbb{Z}$ satisfy $\tau_i \tau_j \tau_i^{-1} = \tau_{2i-j}$. This is the sense in which it is a "coloring": if in a knot diagram we place elements of $\mathbb{Z}/n\mathbb{Z}$ ("colors") on each arc so that at each crossing the elements satisfy $2i = j + k$, with i the element for the overstrand and j, k the elements for the two incident understrands, then by the Wirtinger presentation there is such a homomorphism to D_n.

Instead of coloring arcs of a diagram, we may also n-color regions. A *region n-coloring* is an assignment of values of $\mathbb{Z}/n\mathbb{Z}$ such that at each crossing

the coloring satisfies $A + B = C + D$. Letting $x = A + B$, $y = B + C$, and $z = A + D$, then $2x = y + z$. Given an edge coloring with a choice of color for the "outer" region, the remaining regions can be uniquely colored. We will use the convention that the "outer" region is colored with 0. A fact about Fox n-colorings is that they form an abelian group under arc-wise addition, so by adding a constant coloring to a given coloring we can assume any particular arc has any color we wish. In particular, there is a Fox n-coloring such that in the corresponding region n-coloring both the starred regions are colored with 0.

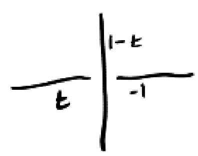
At $t = -1$, the $t, t, 1, 1$ Alexander weights are $-1, -1, 1, 1$, which for the above crossing corresponds to the equation $A+B = C+D$. The determinant $|\Delta_K(-1)|$ is the determinant of the matrix for this system of equations, with the columns for the starred regions removed (corresponding to those regions being assigned the color 0), and the determinant is an obstruction to the existence of a non-trivial solution:

Theorem 6.4. *Nontrivial Fox n-colorings exist iff n divides* $|\Delta_K(-1)|$.

Proof. If n divides $|\Delta_K(-1)|$, then the determinant vanishes modulo n, giving nontrivial region colorings, and therefore nontrivial Fox n-colorings. □

Example 6.5. Every link diagram's regions can be 2-colored so that every pair of neighboring regions is given a different color. A *checkerboard graph* (or *Tait graph*) is a graph formed by taking all regions of the same color as vertices then connecting pairs of vertices by an edge if they are across from each other through a crossing. If K is an alternating knot with an alternating diagram, then $|\Delta_K(-1)|$ is also the number of maximal trees in a checkerboard graph. This comes from seeing that at $t = -1$ each state contributes the same sign in the state sum.

6.1. **Another state sum.** Instead of using the Dehn presentation, we can also use the Wirtinger presentation.[6] Let us consider the exact presentation from Remark 3.28. Indicate on a knot diagram the removal of a row and column by putting a star on a crossing and on an arc. For each crossing, we place the weights on the three arcs at that crossing as follows:



A *marker* is a dot drawn on an arc near a crossing, and a *state* is a collection of markers on unstarred arcs such that every unstarred crossing has exactly one nearby marker. (From the crossing's point of view, a marker is placed on one of the three incident arcs, and in a state each unstarred arc receives exactly one marker.) If we star a crossing and its overstrand arc, then we can place the star as if it were a marker in a state. To a state, we associate a value which is the product of all weights at the markers.

In the expansion of the determinant for the Alexander polynomial, each *state* is a subset of unstarred arcs such that for each unstarred crossing, only one of its incident arcs is selected (chosen arcs are allowed to be incident to a crossing twice, like in a Reidemeister I loop). We can indicate a state by drawing dots on arcs.

7. The Burau representation

8. Vassiliev invariants

9. The Alexander quandle

sec:quandles

This section is not complete.

[6]Warning: this does not seem to go very far, which might be why I have not seen it done.

Quandles are an algebraic structure that were introduced by Joyce in [17] to describe a complete knot invariant (up to orientation-reversing mirror image). The structure had been independently discovered by Takasaki, Conway–Wraith, and Matveev either before or contemporaneously.

Let X be a set acted upon by a group G. An *(augmented) quandle* is a function $\lambda : X \to G$ such that, for all $x \in X$ and $g \in G$,

(1) $g\lambda_x g^{-1} = \lambda_{gx}$, and
(2) $\lambda_x x = x$.

For a *rack*, the second axiom is omitted. If G is generated by $\lambda(X)$, then this matches the usual definition of a quandle, where the first axiom can be written as $\lambda_y\lambda_x = \lambda_{\lambda_y x}\lambda_y$. Or, with the notation $y \triangleleft x = \lambda_y x$ ("x under y"), then, for all $w, x, y \in X$, with $g = \lambda_y$ we have

$$y \triangleleft (x \triangleleft w) = (y \triangleleft x) \triangleleft (y \triangleleft w),$$

displaying $\triangleleft$ as a "self-distributive" binary operation. Quandles are sometimes called the "algebra of knots," and the axioms can be represented graphically as in Figure 9, in anticipation of Definition 9.2.

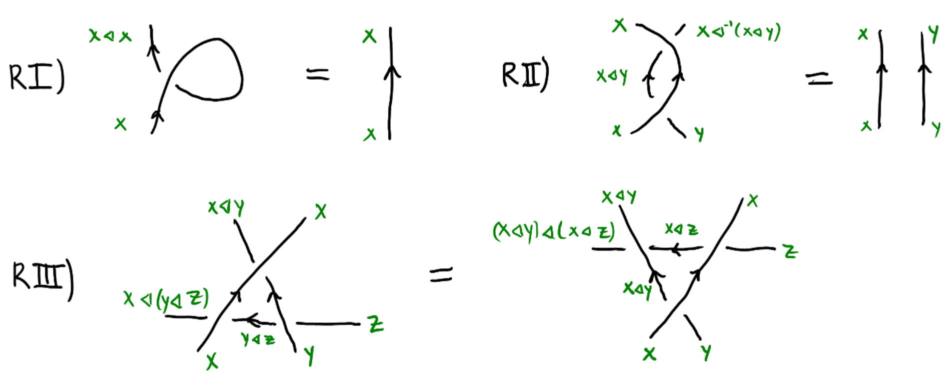


Figure 9. Quandle axioms portrayed as Reidemeister moves. The second Reidemeister move represents the group action. fig:quandle-reidemeister

A homomorphism between quandles $\lambda : X \to G$ and $\mu : Y \to H$ is a function $f : X \to Y$ and a homomorphism $F : G \to H$ such that $F(\lambda_x) = \mu_{f(x)}$ and $f(gy) = F(g)f(y)$. If G and H are generated by $\lambda(X)$ and $\mu(Y)$, respectively, then all that needs to be checked is that $f(y \triangleleft x) = f(y) \triangleleft f(x)$.

Every group G has an associated *conjugation quandle* $c : \operatorname{conj}(G) \to G$ given by $\operatorname{conj}(G) = G$ and $c_g h = ghg^{-1}$. Quandles have a natural homomorphism to the conjugation quandle of their underlying group. The universal conjugation quandle for a quandle X is the conjugation quandle of the associated group $\operatorname{Adconj}(X) = \langle \{g_x : x \in X\} \mid g_x g_y g_x^{-1} = g_{\lambda_x y}\rangle$ that factors $X \to \operatorname{conj}(G)$ by $x \mapsto g_x$ and $g_x \mapsto \lambda_x$.

Example 9.1. Quandle structures show up in some familiar places.

- A group acting on itself by conjugation is $\operatorname{conj}(G)$.
- For G a Lie group and $\mathfrak{g}$ its Lie algebra, the composition of $\exp : \mathfrak{g} \to G$ and $\operatorname{Ad} : G \to \operatorname{Aut}\mathfrak{g}$ forms a quandle $\mathfrak{g} \to \operatorname{Aut}\mathfrak{g}$. The second axiom is that $\operatorname{Ad}_{\exp(x)} x = \exp(\operatorname{ad}_x)x = x$.

- A Riemannian manifold X is a *symmetric space* if it is connected, if $\mathrm{Isom}(X)$ acts on X transitively, and if there is an isometry Φ such that $\Phi^2 = \mathrm{id}_X$ and Φ has at least one isolated fixed point; by translations, there is then such a Φ_x for each $x \in X$ with $\Phi_x x = x$. (Every such space is isometric to G/K for some connected Lie group G with a compact subgroup K, where Φ corresponds to an involutive automorphism of G for which K is an open subgroup of its centralizer in G.) An example is the n-sphere S^n with $\mathrm{Isom}(S^n) = O(n+1)$ and Φ_x being reflection through x, which is a matrix that is similar to
$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}.$$
The function $X \to \mathrm{Isom}(X)$ given by $x \mapsto \Phi_x$ forms a quandle.

def:fundamental-quandle

Definition 9.2. Let L be an oriented link, let $T = \partial\overline{\nu(L)}$ be the boundary of a closed tubular neighborhood of L, and let $* \in S^3 - \nu(L)$ be a basepoint. Let $\mathcal{Q}(L,*)$ be the set of all paths $[0,1] \to S^3 - \nu(L)$ from $*$ to T up to homotopy (allowing the endpoint to roam freely on T). The group $\pi_1(S^3 - L, *)$ acts on $\mathcal{Q}(L,*)$ by the usual concatenation of paths. For each point $x \in T$, let $\mu_x \subset T$ be a meridian loop with basepoint x that has linking number $+1$ with L. Define $\lambda : \mathcal{Q}(L,*) \to \pi_1(S^3 - L, *)$ by $[p] \mapsto [p \cdot \overline{\mu_x} \cdot \overline{p}]$, where overline means to reverse a path (see Figure 10). This is the *fundamental quandle* of L.

FIGURE 10. The quandle operation for the fundamental quandle constructs the path that is on the other side of the crossing, in the overstrand's meridian direction.

fig:quandle-knot-op

Remark 9.3. If we take a diagram for a link L, then if we take $*$ to be a point above the diagram, $\mathcal{Q}(L,*)$ is generated by straight-line paths from $*$ to each overstrand. Notice that the relations in $\mathrm{Adconj}(\mathcal{Q}(K))$ then include all the Wirtinger relations, so the associated group is isomorphic to $\pi_1(L,*)$. Hence, $\mathcal{Q}(L,*)$ remembers $\pi_1(L,*)$ even if we forget the particular group action by passing to the quandle $\mathcal{Q}(L,*) \to \mathrm{Adconj}(\mathcal{Q}(K))$.

Remark 9.4. The *fundamental rack* for a framed oriented link L is instead paths from $*$ to a system of longitudinal curves on T.

Theorem 9.5. *For K an oriented knot, $\mathcal{Q}(K,*)$ is a complete invariant of K up to orientation-reversing mirror image.*

Proof. As discussed in the remark, from the quandle we can recover $G = \pi_1(K,*)$. Let $[p] \in \mathcal{Q}(K,*)$ and $H = \mathrm{Stab}_G(p)$. Let $T = \partial\overline{\nu(K)}$ as in the definition of the fundamental quandle.

- For $g \in \pi_1(T, p(1))$, the path $(p \cdot g \cdot \overline{p}) \cdot p$ is homotopic to p. Hence $\pi_1(T \cup p, *) \subset H$.
- For $g \in H$, since $g \cdot p \sim p$, let h be a path from $gp(1)$ to $p(1)$ over the homotopy. Then $g = p\overline{h}\overline{p} \in \pi_1(T \cup p, *)$.

Thus the stabilizer H is a peripheral subgroup.

Given a knot K' with an isomorphic knot quandle, we obtain an isomorphism $\pi_1(S^3 - K) \to \pi_1(S^3 - K')$ carrying H to a peripheral subgroup and a meridian to a meridian. Using the fact that knot complements are Eilenberg-MacLane spaces, there is a homotopy equivalence $S^3 - K \to S^3 - K'$ that induces this isomorphism. Waldhausen's theorem [33] applies, and one can conclude this homotopy equivalence is homotopic to a homeomorphism that is a restriction of a homeomorphism $S^3 \to S^3$. Since the unoriented mapping class group of S^3 is $\mathbb{Z}/2\mathbb{Z}$, then K is isotopic to either K' or the orientation-reversed mirror image of K'. □

Remark 9.6. This works for non-split links as well. Consider the equivalence relation on $\mathcal{Q}(L)$ generated by $p_1 \sim p_2$ if $p_1 = \lambda_q p_2$ for some $q \in \mathcal{Q}(L)$. (A quandle is called *algebraically connected* if there is exactly one equivalence class.) The equivalence classes are in one-to-one correspondence with components of L. If we take one representative per equivalence class, then the corresponding collection of stabilizers gives a system of peripheral subgroups and meridians for $S^3 - \nu(L)$, and, supposing L is non-split, the above argument applies to show $Q(L)$ characterizes L up to orientation-reversed mirror image.

Remark 9.7. Fenn and Rourke show in [10] that the fundamental rack of a non-split framed link is a complete invariant up to orientation-reversing mirror image.

remark:quandle-group-form

Remark 9.8. Observe that if we choose $*$ to be a point on the boundary torus T itself then elements of $\mathcal{Q}(K,*)$ are equivalent to elements of $\pi_1(S^3 - \nu(K),*)/\pi_1(T,*)$, cosets of the peripheral subgroup. If we take the canonical meridian $\mu \in \pi_1(T,*)$, then the map $\mathcal{Q}(K,*) \to \pi_1(S^3 - \nu(K),*)$ as a map $\lambda : \pi_1(S^3 - \nu(K),*)/\pi_1(T,*) \to \pi_1(S^3 - \nu(K),*)$ is $p \mapsto p\mu^{-1}p^{-1}$, which is well-defined because $\pi_1(T,*)$ is abelian. Then $g\lambda_p g^{-1} = gp\mu^{-1}p^{-1}g^{-1} = \lambda_{gp}$ and $\lambda_p p = p\mu^{-1}p^{-1}p = p\mu^{-1} = p$ (since the elements are cosets of $\pi_1(T,*)$).

Definition 9.9. Let A be a $\mathbb{Z}[t^{\pm 1}]$-module, and let $\mathrm{Aff}(A) := \mathrm{Aut}_{\mathbb{Z}[t^{\pm 1}]}(A) \ltimes A$, where $(x,a) \in \mathrm{Aff}(A)$ corresponds to the affine transformation $b \mapsto xb + a$. An *Alexander quandle* is a quandle $\alpha : A \to \mathrm{Aff}(A)$ defined by $\alpha_a(b) = t^{-1}b + (1 - t^{-1})a$.[7]

Consider $\mathbb{Z}[t^{\pm 1}]$ as an Alexander quandle, and suppose there is a quandle homomorphism $f : \mathcal{Q}(K) \to \mathbb{Z}[t^{\pm 1}]$. Then $f(p\mu^{-1}p^{-1}q) = f(\lambda_p q) = t^{-1}f(q) + (1 - t^{-1})f(p)$.

[7] This is often instead $\alpha_a(b) = tb + (1-t)a$, but t^{-1} matches our meridian convention better.

Given a quandle (Q,λ), there is an associated $Z[t^{\pm}]$-module $A(Q)$ given by

$$A(Q) := \mathbb{Z}[t^{\pm 1}]\langle Q\rangle/(\lambda_a b = t^{-1}b + (1-t^{-1})a \text{ for all } a,b\in Q)$$

for which λ extends to the quandle operation of $A(Q)$ as an Alexander quandle. The map $Q\to A(Q)$ given by $a\mapsto a$ is a quandle homomorphism. Every quandle homomorphism from Q to an Alexander quandle factors through $A(Q)$.

The associated quandle $A(\mathcal{Q}(K))$ is a sort of linearization of $\mathcal{Q}(K)$ of a type we have seen before. Regard $\mathcal{Q}(K)$ from the point of view of Remark 9.8. If we take ∂_2 of $\lambda_a(b) = a\mu^{-1}a^{-1}b$, we get $\lambda_a(b) = a - t^{-1}\mu - t^{-1}a + t^{-1}b = t^{-1}b + (1-t^{-1})a - t^{-1}\mu$.

$\lambda_{\lambda_a b}\lambda_c = \lambda_{\lambda_a c}\lambda_b$

9.1. **Projective Alexander quandles.** Consider a field F that is a $\mathbb{Z}[t^{\pm 1}]$-module, for example $\mathbb{Q}(t)$, $\mathbb{C}(t)$, or $\mathbb{C}$ with t acting by some $c\in\mathbb{C}$.

The space $F\mathbb{P}^1$ of homogeneous points $[a,b]$

10. Reidemeister torsion

The following account of Reidemeister torsion largely comes from [21, Section 8]. Consider a pair (K,L) of finite CW complexes with K connected. Choose a homomorphism $h:\pi_1(K)\to F^\times$ for F a field. This gives rise to a homomorphism $\mathbb{Z}[\pi_1(K)]\to F^\times$ that factors through $\mathbb{Z}[H_1(K)]$. Let $(\hat{K},\hat{L})$ denote the universal cover of (K,L). We can form the chain complex $C'_\bullet = F\otimes_{\mathbb{Z}[\pi_1(K)]} C_\bullet(\hat{K},\hat{L})$ using the action of $\pi_1(K)$ on $(\hat{X},\hat{L})$ by deck transformations. The torsion $\tau(C'_\bullet)$ is an element of $F^\times/\{\pm 1\}$ (an element of $\overline{K}_1F$) up to multiplication by $h(\pi_1(K))$. This $\tau(K,L)\in F^\times/\pm h(\pi_1(K))$ is the *Reidemeister torsion*.

For a knot complement, $\tau(S^3-\nu(K)) = (1-t)/\Delta_K(t)$ for $F=\mathbb{Q}(t)$.

11. Knot Floer homology

References

[1] J. W. Alexander, *Topological invariants of knots and links*, Transactions of the American Mathematical Society **30** (1928), no. 2, 275–306. MR 1501429

[2] J. C. Cha and C. Livingston, *Knotinfo: Table of knot invariants*, http://www.indiana.edu/~knotinfo.

[3] Tim D. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398, arXiv:math/0206258v2 [math.GT]. MR 2077670

[4] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 329–358. MR 0258014

[5] R. H. Crowell, *The group G'/G'' of a knot group G*, Duke Mathematical Journal **30** (1963), 349–354. MR 0154277

[6] ______, *The annihilator of a knot module*, Proc. Amer. Math. Soc. **15** (1964), 696–700. MR 0167976

[7] Richard H. Crowell and Ralph H. Fox, *Introduction to knot theory*, Based upon lectures given at Haverford College under the Philips Lecture Program, Ginn and Co., Boston, Mass., 1963. MR 0146828

[8] M. Dehn, *Die beiden Kleeblattschlingen*, Mathematische Annalen **75** (1914), no. 3, 402–413. MR 1511799

[9] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960

[10] Roger Fenn and Colin Rourke, *Racks and links in codimension two*, Journal of Knot Theory and its Ramifications **1** (1992), no. 4, 343–406. MR 1194995

[11] R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 120–167. MR 0140099

[12] R. H. Fox and N. Smythe, *An ideal class invariant of knots*, Proc. Amer. Math. Soc. **15** (1964), 707–709. MR 0165516
[13] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 239–246. MR 776477
[14] C. McA. Gordon, *Some aspects of classical knot theory*, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), Lecture Notes in Math., vol. 685, Springer, Berlin, 1978, pp. 1–60. MR 521730
[15] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354
[16] Vaughan F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111. MR 766964
[17] David Joyce, *A classifying invariant of knots, the knot quandle*, Journal of Pure and Applied Algebra **23** (1982), no. 1, 37–65.
[18] Louis H. Kauffman, *Remarks on formal knot theory*, arXiv:math/0605622v1 [math.GT].
[19] ———, *Formal knot theory*, Mathematical Notes, vol. 30, Princeton University Press, Princeton, NJ, 1983. MR 712133
[20] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978
[21] J. Milnor, *Whitehead torsion*, Bulletin of the American Mathematical Society **72** (1966), 358–426. MR 0196736
[22] John W. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 115–133. MR 0242163
[23] Józef H. Przytycki and Paweł Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1988), no. 2, 115–139. MR 945888
[24] Dale Rolfsen, *Knots and links*, Publish or Perish, Inc., Berkeley, Calif., 1976, Mathematics Lecture Series, No. 7. MR 0515288
[25] Antonio Sartori, *The Alexander polynomial as quantum invariant of links*, Ark. Mat. **53** (2015), no. 1, 177–202. MR 3319619
[26] Daniel S. Silver and Susan G. Williams, *On modules over Laurent polynomial rings*, arXiv:1006.4153v2 [math.AC].
[27] John Stallings, *On fibering certain* 3*-manifolds*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 95–100. MR 0158375
[28] Alexander Stoimenow, *Knot data tables*, `http://stoimenov.net/stoimeno/homepage/ptab`.
[29] D. W. Sumners, *H_2 of the commutator subgroup of a knot group*, Proc. Amer. Math. Soc. **28** (1971), 319–320. MR 0275416
[30] G. Torres and R. H. Fox, *Dual presentations of the group of a knot*, Annals of Mathematics. Second Series **59** (1954), 211–218. MR 0062439
[31] V. G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988), no. 3, 527–553. MR 939474
[32] O. Ya. Viro, *Quantum relatives of the Alexander polynomial*, Algebra i Analiz **18** (2006), no. 3, 63–157. MR 2255851
[33] Friedhelm Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR 0224099

Department of Mathematics, University of California, Berkeley, California 94720-3840
Email address: `kmill@math.berkeley.edu`