Quiz 5

1. (5 points) A game begins by distributing five different cards to each of the three players. How many ways are there for the game to begin if each player must receive at least one card?

This is asking for the number of onto functions from a set of five objects (the cards) onto a set of three objects (the players). The onto-ness is what guarantees each player receives at least one card. A formula from the book, derived from the inclusion-exclusion principle, gives

$$3^5 - \binom{3}{2}2^5 + \binom{3}{1}1^5 = 243 - 96 + 3 = 150$$

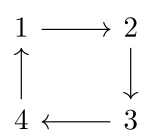
This is the number of functions, minus the number of functions which miss at least one element in the image (range), plus the number of functions which miss at least two elements in the image.

2. (2 points) Come up with a relation which is transitive and symmetric but not reflexive.

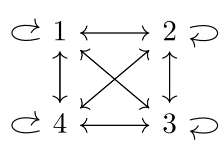
The inspiration for this problem is that if aRb then by symmetry bRa and then by transitivity aRa, so in a symmetric and transitive relation, any element which is related to something is related to itself. This suggests that all we need is a relation where there is something not related to anything. There are infinitely many solutions to this, but the simplest is $A = \{1\}$ and $R = \{\}$. This is vacuously transitive and symmetric.

3. (3 points) Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. (a) Draw a directed graph representing R. (b) Draw a directed graph representing the transitive closure for R.

(a)



(b) Transitive closure is that whenever there is a path from a to b then there is an edge from a to b. Since we can get to any vertex starting from any vertex, the transitive closure contains all possible edges: (and don't forget the self-loops!)



(For fun #1) For a relation $R \subseteq A \times B$, show that $R^{-1} \circ R$ is an equivalence relation on A.

This problem is not correct. (1) Like in 3(a), the relation might not end up being reflexive unless R is "cosurjective." (2) It might not be transitive, for instance if $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $R = \{(1, a), (2, a), (2, b), (3, b)\}$. However, $R^{-1} \circ R$ definitely will be symmetric. I think the proper condition is for R to be a function (cosurjective implies reflexive, coinjective implies transitive).