Quiz 4

1. (5 points) Thomas commutes to Berkeley every day, half the time by car and half the time by BART. When Thomas drives, there is a 50% chance he arrives late, and when he uses BART, there is a 10% chance he arrives late. One day, you observe Thomas arriving late. What is the probability he commuted by car?

Let C be the event Thomas commutes by car, and let L be the event Thomas arrives late. The given data are

$$p(C) = 1/2 \qquad p(L \mid C) = 1/2$$
$$p(\overline{C}) = 1/2 \qquad p(L \mid \overline{C}) = 1/10$$

The question is asking for the value of $p(C \mid L)$. This is a run-of-the-mill application of Bayes's rule, which is

$$p(C \mid L) = \frac{p(L \mid C)p(C)}{p(L)}$$

We have $p(L) = p(L \mid C)p(C) + p(L \mid \overline{C})p(\overline{C}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{10} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{6}{5}$, so

$$p(C \mid L) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{6}{5}} = \frac{5}{6}.$$

2. (5 points) There is a game where you roll three six-sided fair dice one at a time, and you lose if you ever roll a number less than the previous roll. What is the probability you will win?

There are 6^3 possible sequences of three rolls, so if m represents the number of ways to win, $\frac{m}{6^3}$ will be the winning probability.
It is possible to exhaustively consider each winning case, but here we will give a short solution. If r_1, r_2, r_3 represent the values of the three rolls, we want to count the number of solutions to $1 \le r_1 \le r_2 \le r_3 \le 6$, since these are exactly the winning sequences. A trick is to let $a = r_1 - 1$, $b = r_2 - r_1$, and $c = r_3 - r_2$, so then solving $a + b + c + d = 5$ for $a, b, c, d \ge 0$ gives winning sequences $r_1 = 1 + a$, $r_2 = 1 + a + b$, and $r_3 = 1 + a + b + c$. The number of ways to solve this equation is given by "stars and bars" with 5 stars and 3 bars, so $m = \binom{8}{3}$.
Without the d trick, we instead solve $a + b + c \le 5$, so we can do "stars and bars" for 0 through 5 stars and 2 bars:

$$m = \sum_{k=0}^{5} \binom{k+2}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2}$$

(by the hockey-stick identity, this is just $\binom{8}{3}$).
Thus, the probability is

$$\frac{\binom{8}{3}}{6^3} = \frac{7}{27}$$