Quiz 3

1. (5 points) How many 0's are at the end of the base-10 representation of $100! = 1 \cdot 2 \cdot 3 \cdots 99 \cdot 100$?

The number of zeros at the end of a number is the number of times it can be evenly divided by 10. Since $10 = 2 \cdot 5$, this means we can find how many factors of 2 and how many factors of 5 are present in the prime factorization of the number, and the "limiting reagent" between the two governs how many 0's there are. We can determine how many 2's and 5's are in the prime factorization of 100! by seeing how many 2's and 5's each of the numbers $1, 2, \ldots, 100$ contribute.

Number of 2's in the prime factorization. The following is possibly confusing explanation. For sake of notation, let n_k be the number of numbers in $1, 2, \ldots, 100$ which are a multiple of 2^k but not a multiple of 2^{k+1}, so for instance n_1 counts the multiples of two which are not multiples of four. The number of twos in the prime factorization is then $n_1 + 2n_2 + 3n_3 + \cdots$, since, for instance, a number which is multiple of $8 = 2^3$ but not $16 = 2^4$ gives 3 twos. To avoid any inclusion-exclusion, we can rewrite this as $(n_1 + n_2 + n_3 + \cdots) + (n_2 + n_3 + n_4 + \cdots) + (n_3 + n_4 + n_5 + \cdots) + \cdots$. If m_k is the number of numbers in $1, 2, \ldots, 100$ which are a multiple of 2^k, then this is just $m_1 + m_2 + m_3 + \cdots$. We could also have started from this because, for instance, a number which has three 2's in its prime factorization will be counted by m_1, m_2, and m_3, and by only those, and so this sum counts those three 2's. The number of multiples of a number is just give by division, rounding down: $m_k = \lfloor \frac{100}{2^k} \rfloor$. Hence, the number of 2's in the prime factorization of 100! is

$$\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor + \left\lfloor \frac{100}{16} \right\rfloor + \left\lfloor \frac{100}{32} \right\rfloor + \left\lfloor \frac{100}{64} \right\rfloor + \left\lfloor \frac{100}{128} \right\rfloor = 50 + 25 + 12 + 6 + 3 + 1 + 0$$

Let's just say this is more than 50.

Number of 5's in the prime factorization. Same idea:

$$\left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor + \left\lfloor \frac{100}{125} \right\rfloor = 20 + 4 + 0 = 24$$

Therefore, 100! has $\boxed{24}$ zeros at the end since there are exactly 24 fives in the prime factorization and at least that many 2's.

2. (5 points) How many length-6 strings with alphabet $\{a, b, c\}$ have two a's or three b's (or both)?

The problem was intended to mean "*exactly* two a's or exactly three b's". Let us use inclusion-exclusion:

$$\#\{\text{two } a\text{'s or three } b\text{'s}\} = \#\{\text{two } a\text{'s}\} + \#\{\text{three } b\text{'s}\} - \#\{\text{two } a\text{'s and three } b\text{'s}\}.$$

In the first case, a string with two a's can be described by the location of those two a's, with an arbitrary length-4 string of b's and c's filling in the rest, so

$$\#\{\text{two } a\text{'s}\} = \binom{6}{2} 2^4 = 6 \cdot 5 \cdot 2^3 = 24 \cdot 10 = 240$$

Similarly, for the second case a string of three b's can be described by the location of those three b's, along with an arbitrary length-3 string of a's and c's filling in the rest, so

$$\#\{\text{three } b\text{'s}\} = \binom{6}{3}2^3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2}2^3 = 10 \cdot 2^4 = 160$$

Finally, a string of two a's and three b's contains one c. We can describe the location of the c, and among the rest the location of the two a's, with the remaining three locations filled by b's, so it is

$$\#\{\text{two } a\text{'s and three } b\text{'s}\} = \binom{6}{1}\binom{5}{2}\binom{3}{3} = 6 \cdot 10 = 60.$$

Thus, $\#\{\text{two } a\text{'s or three } b\text{'s}\} = 240 + 160 - 60 = \boxed{340}$.

The third case can be handled with the *multinomial coefficient*. The number of strings made of ℓ a's, m b's and n c's is given by $\frac{(\ell+m+n)!}{\ell!m!n!}$, so it is

$$\#\{\text{two } a\text{'s and three } b\text{'s}\} = \frac{6!}{2!3!1!} = \frac{6 \cdot 5 \cdot 4}{2} = 60.$$

(For fun) How many people do you need to guarantee that at least 6 have the same birth month?

The generalized pigeonhole principle is that the maximum is at least the mean. Given n people, what is the mean number of people born per month? That is $\frac{n}{12}$. Since the number of people born any particular month is an integer, the maximum is at least $\lceil \frac{n}{12} \rceil$, so we want to find the least n such that

$$\lceil \frac{n}{12} \rceil \geq 6$$

Using the definition of ceiling, we just need to solve

$$\frac{n}{12} > 5$$

so $n > 60$. That is, $n \geq 61$, since people aren't fractional.