MATH 55. February 8, 2017. (15 minutes, 10 points) Name: SOLUTIONS

Quiz 2

1. (5 points) Let A and B be sets. Prove: $A \subseteq B$ if and only if $A \cap B = A$. *(Please structure the proof in the outline format I demonstrated in class. Hint: start by outlining the "if and only if" and set equality proofs. Every step you fill in is more partial credit.)*

This question was designed so that the most direct proof could work. In excruciating detail, here is one way to give it.

- Claim: $A \subseteq B \to A \cap B = A$.
  - Assume $A \subseteq B$.
    - Claim: $A \cap B \subseteq A$.
      - Let $x \in A \cap B$ be arbitrary.
        - Then by definition of intersection, $x \in A$.
      - Thus, for all $x \in A \cap B$, $x \in A$.
    - Claim: $A \subseteq A \cap B$.
      - Let $x \in A$ be arbitrary.
        - Since $A \subseteq B$, $x \in B$.
        - Since $x \in A$ and $x \in B$, $x \in A \cap B$.
      - Thus, for all $x \in A$, $x \in A \cap B$.
    - Therefore, $A \cap B = A$.
  - Therefore $A \subseteq B \to A \cap B = A$.
- Claim: $A \cap B = A \to A \subseteq B$.
  - Assume $A \cap B = A$.
    - Let $x \in A$ be arbitrary.
      - Since $A = A \cap B$, $x \in A \cap B$.
      - By definition of intersection, $x \in B$.
    - Thus, for all $x \in A$, $x \in B$, hence $A \subseteq B$.
- Therefore, $A \cap B = A \leftrightarrow A \subseteq B$.

(Certain implications may be proved instead by contradiction. Giving a proof of something like $A \cap B \subseteq A$ is usually omitted, since it is "trivial" in the logical sense. The proof is harder to understand than the corresponding Venn diagram, which I invite you to draw.)

2. (5 points) Suppose $f : A \to B$ is a function with the property that $|f(S)| = |S|$ whenever $S \subseteq A$ is a finite set. Prove that f is one-to-one (injective). *(Recall that $f(S)$ means $\{f(x) \mid x \in S\}$.)*

1. Let $f : A \to B$ be an arbitrary such function.

   (a) Let $x, y \in A$ be arbitrary such that $x \neq y$.

      i. Let $S = \{x, y\}$, which is a two-element set.

      ii. $|f(S)| = |S|$, by assumption, and $|S| = 2$.

iii. $f(S) = \{f(x), f(y)\}$ by definition.
iv. Since $|f(S)| = 2$, then $f(x) \neq f(y)$, otherwise $|f(S)|$ would be 1.

(b) Thus for all $x, y \in A$, if $x \neq y$ then $f(x) \neq f(y)$.

(c) This is the definition of f being one-to-one.

2. Therefore, for all f satisfying the given condition, f is one-to-one.

(Fun fact #1) The set $A = \{f \mid f : \mathbb{N} \to \mathbb{N} \text{ and for all } n \in \mathbb{N},\ f(n) \leq f(n+1)\}$ is uncountable.
(Fun fact #2) The set $B = \{f \mid f : \mathbb{N} \to \mathbb{N} \text{ and for all } n \in \mathbb{N},\ f(n) \geq f(n+1)\}$ is countable.

(For fun) Why?

Recall the notation Y^X represents the set of functions $X \to Y$. We can define an injection $F : \mathbb{N}^{\mathbb{N}} \to A$ which takes an arbitrary function to a function which is nondecreasing. For $g : \mathbb{N} \to \mathbb{N}$, $F(g) \in A$ is a nondecreasing function defined by

$$F(g)(n) = \sum_{i=0}^{n} g(i)$$

This is nondecreasing since $F(g)(n+1) = F(g)(n) + g(n+1) \geq F(g)(n)$ follows from $g(n+1) \geq 0$ for all n. This is an injection since if $F(g) = F(g')$, then $\sum_{i=0}^{n} g(i) = \sum_{i=0}^{n} g'(i)$ for all $n \in \mathbb{N}$, and by induction $g(n) = g'(n)$ for all $n \in \mathbb{N}$. (Prove the induction!)
So, an injection $\mathbb{N}^{\mathbb{N}} \to A$ means $|\mathbb{N}^{\mathbb{N}}| \leq |A|$, and since $\mathbb{N}^{\mathbb{N}}$ contains $2^{\mathbb{N}}$ (infinitely long binary sequences) and $2^{\mathbb{N}}$ is uncountable, A is uncountable, too.

The slight change in the definition of B suddenly makes the set countable. This is because any particular sequence in B can only decrease in finitely many indices (since the codomain is $\mathbb{N}$). In particular, knowing, say, that $f(100) = 22$ means the sequence can only decrease at most 22 more times after $n = 100$. This means a sequence can be defined by where it decreases and by how much it decreases at those points; that is, by a finite set of pairs of natural numbers. The set of finite subsets of a countable set is countable, and $\mathbb{N} \times \mathbb{N}$ is countable.

(There are many ways to try to explain that A is uncountable and that B is countable. If this particular way of writing it down didn't help you, please try to find/invent an explanation which convinces you!)