MATH 55. January 25, 2017. (15 minutes, 10 points) Name: SOLUTIONS

Quiz 1

1. (5 points) Let p and q stand for propositions. For each of the following, determine whether or not the proposition is a tautology. (a) $(p \vee q) \to (\neg p \to q)$ (b) $p \wedge (q \vee \neg q)$ (c) $((p \to q) \to p) \to p$.

In each case, we can either construct a truth table or give a proof. For a truth table, a statement is a tautology if no matter the truth values of p and q the proposition is true. And proofs can only result in tautologies (when assuming no extra axioms).

(a) Truth table:

p	q	$p \vee q$	$\neg p \to q$	$(p \vee q) \to (\neg p \to q)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

The statement is always true, so it is a tautology.

Proof: Assume $p \vee q$. This is equivalent to $\neg\neg p \vee q$, which is equivalent to $\neg p \to q$ (using $a \to b \equiv \neg a \vee b$). Thus, $(p \vee q) \to (\neg p \to q)$.

(b) If $p \equiv F$, then the statement is false, so it is not a tautology.

(c) (This is called *Pierce's Law*. It is equivalent to the law of the excluded middle, or double negation elimination.)

If $p \equiv T$, then since $x \to T$ is true no matter the true value of x, the whole statement is true in this case. If $p \equiv F$, then the statement simplifies in this way: $((F \to q) \to F) \to F \equiv (T \to F) \to F \equiv F \to F \equiv T$. Thus, in either case the statement is true. (This argument is effectively a truth table.)

Proof: Assume $(p \to q) \to p$ is true.

Case I. $p \to q$. Then by modus ponens using the assumption, p.

Case II. $\neg(p \to q)$. This is equivalent to $p \wedge \neg q$, so p. (This case assumes $\neg\neg p \equiv p$.)

In either case, p. Thus, $((p \to q) \to p) \to p$.

2. (5 points) Suppose $C(x) =$ "x is a cat" and $Q(x, y) =$ "x is cuter than y".

(a) Express this statement using quantifiers: "There is a cat that is cuter than all other cats."

(b) Express this statement using quantifiers: "There are two cats, each cuter than the other!"

(c) Using quantifiers, give the negation of statement (a). Put it into a form where the symbol $\neg$ only shows up immediately in front of a C or a Q.

(a) $\exists x\ C(x) \wedge (\forall y\ (x \neq y \wedge C(y)) \to Q(x, y))$. (If "other" means "distinct from the first.")

(b) Depending on the interpretation of whether "two" actually means there are two distinct cats, either of the following is possible:

$\exists x \exists y\ C(x) \wedge C(y) \wedge Q(x, y) \wedge Q(y, x)$

$\exists x \exists y\ C(x) \wedge C(y) \wedge x \neq y \wedge Q(x, y) \wedge Q(y, x)$

(c) $\forall x\ C(x) \to (\exists y\ x \neq y \wedge C(y) \wedge \neg Q(x, y))$ ("For every cat there is some other cat it is not cuter than.")

(For fun #1) Let S be the statement "If S is true, then every rainbow has two ends and Leprechauns keep gold under them." Rainbows do not have ends. Is S a proposition?

Propositions must be either true or false (and not both). If S is true, then S says that every rainbow has two ends, but we agreed rainbows have no ends. But if S is false, then its negation is true, which is "S is true and it is not the case that every rainbow has two ends and that Leprechauns keep gold under them," contradicting the fact S is false.

In general, if you manage to find a statement S which says "$S \to P$", and if you somehow manage to prove that it is actually a proposition, then P is true. This is known as Löb's paradox.

(For fun #2) Express this statement using quantifiers: "There are exactly two distinct cats."

We have to meditate on the meaning of the number two. How do we know when we have at least two things? Simple: there exist two things which are not equal. How do we know when we have no more than two things? One way is to say that if you're ever referring to three things, then two of them must be the same. (If you have two brands of similar looking socks, you don't have to match them ahead of time: just take three socks, and among them you'll have a pair.)

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\begin{aligned}
&(\exists x \exists y (x \neq y \wedge C(x) \wedge C(y))) \\
&\wedge (\forall x \forall y \forall z \ (C(x) \wedge C(y) \wedge C(z)) \to (x = y \vee x = z \vee y = z))
\end{aligned}
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