MATH 54. August 5, 2016. (20 minutes, 10 points max.)   Name:

Quiz 6

1. (5 points) Consider the vector functions $\begin{pmatrix}1\\t\\0\end{pmatrix}$, $\begin{pmatrix}t\\0\\1\end{pmatrix}$, and $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

(a) Are these three vector functions linearly independent on $(-\infty, \infty)$?

One way to check independence is to find some particular value of t where the vector functions values there are linearly independent. At $t = 1$, the vectors are easily shown to be independent by row reduction.
Another way is to compute the Wronskian:

$$\det\begin{pmatrix}1 & t & 0\\ t & 0 & 0\\ 0 & 1 & 1\end{pmatrix} = \det 1 \begin{pmatrix}1 & t\\ t & 0\end{pmatrix}$$
$$= -t^2.$$

This is nonzero when t is, for example 1. All that is needed for independence is for the Wronskian to be nonzero somewhere.

(b) Are these three vector functions solutions to the same homogenous linear system?

No: the Wronskian is zero at $t = 0$. The Wronskian is *never* zero for linearly independent solutions to the same homogeneous linear system.
Alternatively, at $t = 0$, the three vector functions have linearly dependent initial conditions. If they were solutions to the same differential equation, by the uniqueness theorem they would be linearly dependent (which they aren't, by (a)).

2. (5 points) For $\vec{x}'(t) = \begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}\vec{x}(t)$.

(a) Give the general solution to this differential equation.

This is a homogeneous system, so we first find the characteristic polynomial:

$$\det\begin{pmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

so the eigenvalues are 4 and -1. Let us find the eigenvectors. For $\lambda = 4$:

$$\text{Nul}\begin{pmatrix}1-4 & 2\\ 3 & 2-4\end{pmatrix} = \text{Nul}\begin{pmatrix}-3 & 2\\ 3 & -2\end{pmatrix} = \text{Nul}\begin{pmatrix}-3 & 2\\ 0 & 0\end{pmatrix}$$

which has the basis $\left\{\begin{pmatrix}2\\3\end{pmatrix}\right\}$. For the eigenvalue $\lambda = -1$,

$$\text{Nul}\begin{pmatrix}1+1 & 2\\ 3 & 2+1\end{pmatrix} = \text{Nul}\begin{pmatrix}2 & 2\\ 3 & 3\end{pmatrix} = \text{Nul}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$

which has the basis $\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$. We have obtained two eigenvectors (so the matrix is diagonalizable) which means the general solution to the system can be given in the following form:

$$\vec{x}(t) = c_1\begin{pmatrix}2\\3\end{pmatrix}e^{4t} + c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-t}$$

(b) Give the solution with the initial condition $\vec{x}(0) = \begin{pmatrix}0\\5\end{pmatrix}$.

Since

$$\vec{x}(t) = c_1\begin{pmatrix}2\\3\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix},$$

we just need to solve for when this is $\begin{pmatrix}0\\5\end{pmatrix}$. This is a linear system of equations. In this case, the first coordinate must be 0, which means $c_2 = 2c_1$ for a solution, and $c_1 = 1$ appears to work, so

$$\vec{x}(t) = \begin{pmatrix}2\\3\end{pmatrix}e^{4t} + 2\begin{pmatrix}-1\\1\end{pmatrix}e^{-t}$$

satisfies the initial condition.

3. (1 point) Find a fundamental matrix for $\vec{y}'(t) = \begin{pmatrix}0 & 1\\ -1 & 2\end{pmatrix}\vec{y}(t)$. (This corresponds to $y'' - 2y' + y = 0$, which you may use if you wish.)

Option 1. This is the normal form for the mentioned second-order linear differential equation, with $y_1 = y$ and $y_2 = y'$. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2$, so $y = c_1e^t + c_2te^t$. The derivative is $y' = c_1e^t + c_2e^t + c_2te^2$, which means solutions to the linear system are of the form

$$\begin{pmatrix}c_1e^t + c_2te^t\\ c_1e^t + c_2e^t + c_2te^2\end{pmatrix} = \begin{pmatrix}e^t & te^t\\ e^t & e^t + te^2\end{pmatrix}\vec{c}$$

Thus, $\begin{pmatrix}e^t & te^t\\ e^t & e^t + te^t\end{pmatrix}$ is a fundamental matrix.

Option 2. Matrix exponential via generalized eigenspaces. One can check there is only a single eigenvector, so the matrix is not diagonalizable. The generalized eigenspace is $\mathbb{R}^2$ since $\mathrm{Nul}(A-I)^2$ is theoretically two-dimensional. Let $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$ be a basis for it (which I chose just because the first vector is actually an eigenvector). Then,

$$\exp(At)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}e^t$$

since the vector is an eigenvector, and

$$\begin{aligned}\exp(At)\begin{pmatrix}1\\0\end{pmatrix} &= \exp(It)\exp((A-I)t)\begin{pmatrix}1\\0\end{pmatrix}\\ &= e^t\left(\begin{pmatrix}1\\0\end{pmatrix} + t(A-I)\begin{pmatrix}1\\0\end{pmatrix}\right)\\ &= e^t\begin{pmatrix}1-t\\-t\end{pmatrix}\end{aligned}$$

using the formula from the book, utilizing the fact that $(A-I)^2$ times the generalized eigenvector is the zero vector. Putting these together, we get a fundamental matrix

$$\begin{pmatrix}e^t & e^t - te^t\\ e^t & -te^t\end{pmatrix}$$

(notice this is the same as the first fundamental matrix after two column operations).