MATH 54. July 29, 2016. (20 minutes, 10 points max.) Name:

Quiz 5

1. (5 points) Give the general solution to $y'' - 6y' + 9y = e^{-2t}$.

The auxiliary polynomial for the homogeneous equation is $r^2 - 6r + 9 = (r-3)^2$, so the homogeneous solution is of the form $c_1 e^{3t} + c_2 t e^{3t}$. The forcing term corresponds to the root -2, which does not overlap with $3, 3$, so we will use the guess $y_p = ae^{-2t}$ with the method of undetermined coefficients. The derivatives of this are $y_p' = -2ae^{-2t}$ and $y_p'' = 4ae^{-2t}$, and substituting this into $y'' - 6p' + 9y = e^{-2t}$ we get $25ae^{3t} = e^{-2t}$. Thus, for this to be true for all t, $a = \frac{1}{25}$. Adding the homogeneous solution, we get the general solution

$$y = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{25} e^{-2t}.$$

2. (5 points) For $y'' + 2y' + 2y = 0$, find a solution $y(t)$ where $y(0) = 1$ and $y'(0) = 0$.

The auxiliary polynomial is $r^2 + 2r + 2$, which has roots $\frac{-2 \pm \sqrt{2^2 - 4(2)}}{2} = -1 \pm i$. General solutions are of the form $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$.
To get $y(0) = 1$, then $1 = c_1 e^{-0} \cos 0 + c_2 e^{-0} \sin 0 = c_1$. The derivative of $e^{-t} \cos t + c_2 e^{-t} \sin t$ is $-e^{-t} \cos t - e^{-t} \sin t + c_2(-e^{-t} \sin t + e^{-t} \cos t)$, so $y'(0) = -1 + c_2$. Since $y'(0) = 0$, then $c_2 = 1$. Therefore, the solution which satisfies the given initial condition is $y = e^{-t} \cos t + e^{-t} \sin t$. (And we can easily check that this indeed satisfies the initial conditions.)

3. (1 point) Compute the general solution to $y'' + y' + y = e^{-t/2} \cos(\sqrt{\frac{3}{4}} t)$. What happens as $t \to \infty$? (Is this surprising? Think about $y'' + y = \cos(t)$.)

(I realize I meant to say compute the *form* of the general solution.) The auxiliary polynomial is $r^2 + r + 1$, which has roots $\frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \sqrt{\frac{3}{4}}$. The homogeneous solution is then $e^{-t/2}(c_1 \cos(\sqrt{3/4} t) + c_2 \sin(\sqrt{3/4} t))$. Since the forcing term is one of these, the particular solution must be of the form $y_p = te^{-t/2}(a \cos(\sqrt{3/4} t) + b \sin(\sqrt{3/4} t))$. To reduce complexity, we will compute derivatives of different parts of y_p separately.

$$\frac{d}{dt} te^{-t/2} \cos(\sqrt{3/4} t) = e^{-t/2} \cos(\sqrt{3/4} t) - \frac{1}{2} te^{-t/2} \cos(\sqrt{3/4} t) - \sqrt{3/4} te^{-t/2} \sin(\sqrt{3/4} t)$$
$$\frac{d}{dt} te^{-t/2} \sin(\sqrt{3/4} t) = e^{-t/2} \sin(\sqrt{3/4} t) - \frac{1}{2} te^{-t/2} \sin(\sqrt{3/4} t) + \sqrt{3/4} te^{-t/2} \cos(\sqrt{3/4} t)$$
$$\frac{d}{dt} e^{-t/2} \cos(\sqrt{3/4} t) = -\frac{1}{2} e^{-t/2} \cos(\sqrt{3/4} t) - \sqrt{3/4} e^{-t/2} \sin(\sqrt{3/4} t)$$
$$\frac{d}{dt} e^{-t/2} \sin(\sqrt{3/4} t) = -\frac{1}{2} e^{-t/2} \sin(\sqrt{3/4} t) + \sqrt{3/4} e^{-t/2} \cos(\sqrt{3/4} t)$$

This isn't necessary, but one way to organize the computation is to us the basis $\mathcal{B} = \{te^{-t/2}\cos(\sqrt{3/4}t), te^{-t/2}\sin(\sqrt{3/4}t), e^{-t/2}\cos(\sqrt{3/4}t), e^{-t/2}\sin(\sqrt{3/4}t)\}$ and write the matrix of $\frac{d}{dt}$ with respect to it. We get

$$A = \left[\frac{d}{dt}\right]_{\mathcal{B}} = \begin{pmatrix} -1/2 & \sqrt{3/4} & 0 & 0 \\ -\sqrt{3/4} & -1/2 & 0 & 0 \\ 1 & 0 & -1/2 & \sqrt{3/4} \\ 0 & 1 & -\sqrt{3/4} & -1/2 \end{pmatrix}$$

Then, since y_p in this basis is the vector $(a, b, 0, 0)^T$, we just want to solve

$$A^2 \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} + A \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} + I_4 \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

After much computation this simplifies to

$$\begin{pmatrix} 0 \\ 0 \\ b\sqrt{3} \\ -a\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so $b = 1/\sqrt{3}$ and $a = 0$. Thus, the general solution is

$$y = e^{-t/2}(c_1\cos(\sqrt{3/4}t) + c_2\sin(\sqrt{3/4}t)) + \frac{1}{\sqrt{3}}te^{-t/2}\sin(\sqrt{3/4}t)$$

Anyway, I intended to ask only about the *form* of the solution, with which one can answer the next portion.

When $t \to \infty$, both $e^{-t/2} \to 0$ and $te^{-t/2} \to 0$ (the latter because exponentials get smaller faster than a linear function can get bigger). So, $y \to 0$. This is surprising because, unlike the undamped mass-spring system, which driving using one of the homogeneous solutions causes oscillations of unbounded amplitude, the damped mass-spring system has it where drivig it using one of the homogeneous solutions just peters out.