MATH 54. July 15, 2016. (20 minutes, 10 points) Name:

Quiz 4

Make sure to check that $AP = PD$ for diagonalizations.

1. (5 points) Find a complete set of eigenvectors for $\begin{pmatrix} 9 & -3 \\ 14 & -4 \end{pmatrix}$ and specify the eigenvalues. Diagonalize the matrix if it is diagonalizable, otherwise explain why it is not diagonalizable.

First we determine the eigenvalues. The characteristic polynomial is $\left|\begin{smallmatrix} 9-\lambda & -3 \\ 14 & -4-\lambda \end{smallmatrix}\right| = (9-\lambda)(-4-\lambda)+42 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$, so the eigenvalues are 2 and 3.
The eigenspace associated with 2 is $\mathrm{Nul}(A-2I) = \mathrm{Nul}\left(\begin{smallmatrix} 7 & -3 \\ 14 & -6 \end{smallmatrix}\right) = \mathrm{Nul}\left(\begin{smallmatrix} 7 & -3 \\ 0 & 0 \end{smallmatrix}\right)$, which is spanned by $\left(\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right)$.
The eigenspace associated with 3 is $\mathrm{Nul}(A-3I) = \mathrm{Nul}\left(\begin{smallmatrix} 6 & -3 \\ 14 & -7 \end{smallmatrix}\right) = \mathrm{Nul}\left(\begin{smallmatrix} 2 & -1 \\ 0 & 0 \end{smallmatrix}\right)$, which is spanned by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$.
There are two good reasons for the matrix being diagonalizable. The first is that there are two distinct eigenvalues for a matrix with two columns. The second is that the sum of the dimensions of the eigenspaces is two.
Let $P = \left(\begin{smallmatrix} 3 & 1 \\ 7 & 2 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Then the matrix has the diagonalization PDP^{-1}.

2. (5 points) Do the same for $\begin{pmatrix} 3 & 0 & -1 \\ -2 & 4 & -2 \\ -3 & 0 & 1 \end{pmatrix}$.

The characteristic polynomial for this matrix is

$$\begin{aligned}
\begin{vmatrix} 3-\lambda & 0 & -1 \\ -2 & 4-\lambda & -2 \\ -3 & 0 & 1-\lambda \end{vmatrix} &= (4-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -3 & 1-\lambda \end{vmatrix} \\
&= (4-\lambda)((3-\lambda)(1-\lambda)-3) \\
&= (4-\lambda)(3-4\lambda+\lambda^2-3) \\
&= (4-\lambda)^2\lambda
\end{aligned}$$

Thus, the eigenvalues are $0, 4, 4$ with multiplicity.
The eigenspace associated with 0 is

$$\begin{aligned}
\mathrm{Nul}(A-0\lambda) = \mathrm{Nul}\begin{pmatrix} 3 & 0 & -1 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix} &= \mathrm{Nul}\begin{pmatrix} 1 & 4 & -3 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 4 & -3 \\ 0 & 12 & -8 \\ 0 & 0 & 0 \end{pmatrix} \\
= \mathrm{Nul}\begin{pmatrix} 1 & 4 & -3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} &= \mathrm{Nul}\begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}.
\end{aligned}$$

The eigenspace associated with 4 is

$$\mathrm{Nul}(A - 4\lambda) = \mathrm{Nul}\begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -3 & 0 & -3 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Thus, the matrix is diagonalizable because the sum of the dimensions of the eigenspaces is the number of columns of the matrix. So $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

3. (1 point) Show that $\mathrm{rank}\, AB \leq \mathrm{rank}\, A$ and that $\mathrm{rank}\, AB \leq \mathrm{rank}\, B$.

For the first, every vector in the column space of AB is written as $AB\vec{c}$, which is $A(B\vec{c})$, so they are vectors in the column space of A as well. This means $\mathrm{Col}\, AB$ is a subspace of $\mathrm{Col}\, A$, so $\dim \mathrm{Col}\, AB \leq \dim \mathrm{Col}\, A$. Since rank is the dimension of the column space, $\mathrm{rank}\, AB \leq \mathrm{rank}\, A$.

For the second, we apply the first to say $\mathrm{rank}\, B^T A^T \leq \mathrm{rank}\, B^T$. Since $(AB)^T = B^T A^T$ and $\mathrm{rank}\, A^T = \mathrm{rank}\, A$, we obtain $\mathrm{rank}\, AB \leq \mathrm{rank}\, B$.

Alternatively, every vector of the nullspace of B is a vector of the nullspace of AB, so $\dim \mathrm{Nul}\, AB \geq \dim \mathrm{Nul}\, B$. We get the inequality on rank from rank-nullity: $n - \dim \mathrm{Col}\, AB \geq n - \dim \mathrm{Col}\, B$ so $\dim \mathrm{Col}\, AB \leq \dim \mathrm{Col}\, B$.

These are things I noticed while grading:

- If an "eigenspace" is zero-dimensional, that means the number you thought was an eigenvalue is not an eigenvalue. (Check your characteristic polynomial!) Every distinct root of the characteristic polynomial guarantees at least one eigenvector.
- The zero vector is never an eigenvector, by definition.
- If you find n linearly independent eigenvectors for an $n \times n$ matrix, the matrix is diagonalizable.
- It is OK if 0 is an eigenvalue. The matrix can still be diagonalizable.
- It is OK if has an eigenvalue with multiplicity greater than one. The matrix can still be diagonalizable. For instance, I_3 has $1, 1, 1$ as eigenvalues, and it is diagonalizable.
- A diagonalization is writing A as PDP^{-1}. Just say what P and D are equal to. Computing P^{-1} and writing out the matrices P, D, and P^{-1} next to each other is not, technically, a diagonalization.
- If a matrix has a really easy cofactor expansion (for instance, in the second problem), it is usually a good idea to take it.