MATH 54. July 8, 2016. (15 minutes, 10 points) Name:

Quiz 3

1. (5 points) A subspace W of $\mathbb{R}^3$ is spanned by $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$. What is $\dim W$?

Let A be the 3×3 matrix with the given three vectors as columns. $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$, so they are linearly dependent. Since this means the third vector is dependent on the first two, we may remove it and still have a spanning set of W. A test for dependence of two vectors is that neither is a scalar multiple of the other, which is indeed the case for the first two vectors. Thus, the first two vectors are a basis of W, and so $\dim W = 2$.

2. (5 points) A basis for $\mathbb{P}_2$ is $\mathcal{B} = \begin{pmatrix} 1 & 1+x & x^2 \end{pmatrix}$. (a) Find coordinates for $p_1(x) = 2+x+x^2$, $p_2(x) = -x + x^2$, and $p_3(x) = -1 + x + x^2$ with respect to $\mathcal{B}$. (b) Use the coordinates to determine whether these polynomials are linearly independent.

(a) Given a coordinate $(c_1, c_2, c_3) \in \mathbb{R}^3$, the corresponding vector in $\mathbb{P}_2$ is $c_1(1) + c_2(1 + x) + c_3(x^2) = (c_1 + c_2) + c_2 x + c_3 x^2$.
The polynomial $p_1(x)$ needs $c_1 + c_2 = 2$, $c_2 = 1$, and $c_3 = 1$, so its coordinate is $(1, 1, 1)$.
The polynomial $p_2(x)$ needs $c_1 + c_2 = 0$, $c_2 = -1$, and $c_3 = 1$, so its coordinate is $(1, -1, 1)$.
The polynomial $p_3(x)$ needs $c_1 + c_2 = -1$, $c_2 = 1$, and $c_3 = 1$, so its coordinate is $(-2, 1, 1)$.
(b) Whether vectors are linearly independent is whether $c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0$ has only the trivial solution for (c_1, c_2, c_3). Because $\mathcal{B}$ is an isomorphism, we can instead check whether $c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$ has only the trivial solution.
This is a vector equation, which has a corresponding matrix equation which we may row reduce. After row reduction (omitted from this solution), one obtains three pivots, so there is a unique solution: the trivial solution. Thus, the coordinate vectors are independent, and so the three polynomials are independent.

3. (1 point) A 3×3 matrix A satisfies $A^2 = 0$. What are the possible dimensions for $\operatorname{Col} A$ and $\operatorname{Nul} A$? Give them as pairs $(\dim \operatorname{Col} A, \dim \operatorname{Nul} A)$.

Since $A^2 = 0$, for any vector $\vec{v} \in \mathbb{R}^3$, $A(A\vec{v}) = \vec{0}$. Thus, either $A\vec{e}_1 = \vec{0}$, or $A\vec{e}_1$ is nonzero. This means that $\operatorname{Nul} A$ is not the zero subspace, since at least one of $\vec{e}_1$ and $A\vec{e}_1$ are in the nullspace. That is, $\dim \operatorname{Nul} A \neq 0$.
By rank-nullity, the possible dimensions are $(2, 1)$, $(1, 2)$, and $(0, 3)$, since the pairs of dimensions sum to 3, and we exclude $(3, 0)$.

In fact, each are actually possible:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$