MATH 54. July 1, 2016. (15 minutes, 10 points) Name:

Quiz 2

1. (5 points) Find a matrix B satisfying $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 6 & 10 \end{pmatrix}$.

Matrix multiplication AB is defined to be $\begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{pmatrix}$, so for AB to equal a particular matrix C, we can solve for each column of B one at a time using the augmented matrices $(A \mid \vec{c}_1)$, $(A \mid \vec{c}_2)$, and $(A \mid \vec{c}_3)$. (In other words, we need to solve $A\vec{b}_i = \vec{c}_i$ for $i = 1, 2, 3$.)
But, observe that we do not actually need to do row reduction three times! We will do the same operations every time no matter the augmented column, so we may as well just augment the matrix with all three columns of C.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 2 & 1 & -1 & 2 & 5 & 7 \\ 3 & 0 & 1 & 3 & 6 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Thus, $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. One should check that AB is the correct matrix.
Alternatively, you could compute A^{-1} and multiply both sides of the equation by it to derive B.

2. (5 points) Compute the determinant of $\begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{pmatrix}$. Is this matrix invertible?

We could either compute the determinant using cofactor expansion or using row reduction. For cofactor expansion, let us expand along the second row. Thus,

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

Now the third row,

$$\begin{aligned} &= -3\left(5\begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right) \\ &= -3(5(-2(5) - 2(-6)) + 4(-6 - 2(-2))) \\ &= -6. \end{aligned}$$

The matrix has a nonzero determinant, so it is invertible.

Alternatively, we could do row reduction:

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & -17 & 1 \\ 0 & 10 & -21 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & -17 & 1 \\ 0 & 0 & -106 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & -17 & 1 \\ 0 & 0 & -106 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & -17 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & -2 & -17 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -(1)(-2)(3)(-1) = -6.$$

3. (1 point) For what $a, b, c \in \mathbb{R}$ does $\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$ have linearly independent columns? (Please make the condition look as nice as possible, for instance by factoring.)

We can do row reduction to see exactly when there is a pivot in every column, though doing straight row reduction leads to an issue due to having to divide, requiring reasoning that we are not dividing by zero. Instead of this, we can make use of the invertible matrix theorem: a square matrix with linearly independent columns has a nonzero determinant. Let us calculate the determinant with a mixture of row reduction and cofactor expansion:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$\begin{aligned} &= (b-a)(c^2-a^2) - (c-a)(b^2-a^2) \\ &= (b-a)(c-a)((c+a)-(b+a)) \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

This is nonzero exactly when a, b, c are three distinct numbers.