MATH 54. June 24, 2016. (15 minutes, 10 points) Name:

Quiz 1

1. (5 points) Solve $\begin{pmatrix} 1 & 2 & -1 & -1 & -5 \\ 2 & 4 & -1 & 2 & -5 \\ -1 & -2 & 0 & -3 & 0 \end{pmatrix} \vec{x} = \vec{0}$ either in parametric vector form or as a span.

This system has the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 2 & 4 & -1 & 2 & -5 & 0 \\ -1 & -2 & 0 & -3 & 0 & 0 \end{array}\right)$$

which we will put into reduced row echelon form.[a] As always, the order of row operations doesn't matter, though the following is according to the algorithm described by the textbook. First $R_2 - 2R_1 \to R_2$:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ -1 & -2 & 0 & -3 & 0 & 0 \end{array}\right)$$

Then $R_3 + R_1 \to R_3$:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & -1 & -4 & -5 & 0 \end{array}\right)$$

Then $R_3 + R_2 \to R_3$:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is now in row echelon form, and is solvable by back-substitution, but I tend to like putting it into reduced row echelon form. Then $R_1 + R_2 \to R_1$:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Variables x_2, x_4, x_5 are free. We solve $x_1 = -2x_2 - 3x_4$ and $x_3 = -4x_4 - 5x_5$, hence

$$\vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}.$$

The solution set is the span of these three vectors.

[a] In practice, I would just do the reduced row echelon form of the coefficient matrix A, knowing that the row operations will leave the $\vec{0}$ column as $\vec{0}$. This saves some unnecessary writing.

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2. (5 points) Are the columns of $\begin{pmatrix} 1 & -2 & -1 \\ 1 & 0 & 3 \\ 2 & -1 & 4 \end{pmatrix}$ independent? Give a dependence if not.

To determine independence, we see if $A\vec{x} = \vec{0}$ has a non-trivial solution. The non-trivial solution would give us a dependence.
We start with the augmented matrix again and compute reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 1 & 0 & 3 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

First $R_2 - R_1 \to R_2$ then $R_3 - 2R_1 \to R_3$:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right)$$

Then $\frac{1}{2}R_2 \to R_2$:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right)$$

Then $R_3 - 3R_2 \to R_3$:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At this point, we are sure that the columns of the matrix are dependent: fewer pivots than columns implies that there is a non-trivial solution to this homogeneous system.
Finally $R_1 + 2R_2 \to R_1$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We just need *some* non-trivial solution to give a dependence. The free variable is x_3, so we may as well let it equal something simple, like 1 (though you are free to choose your favorite non-zero number). This gives $x_1 = -3$ and $x_2 = -2$. Together, we then have a dependence:

$$\vec{0} = -3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

3. (1 point) For some 3×5 matrix A, the solution set to $A\vec{x} = \vec{0}$ is Span$\{\vec{u}, \vec{v}\}$, with $\vec{u}, \vec{v} \in \mathbb{R}^5$. Do the columns of A span $\mathbb{R}^3$? Give a one-sentence explanation.

The columns of A span $\mathbb{R}^3$ because there is a pivot in each row, which is because A has three pivot columns which is because A has at most two free columns, which is because the homogenous equation's solution set is the span of only two vectors.

In multiple sentences: The solution set is the span of two vectors. This means that A has at most two free columns. Since it has five columns total, this means it has at least three pivot columns. Since A has only three rows, we can see A has exactly one pivot in each row. This is logically equivalent to the columns of A spanning $\mathbb{R}^3$.