1. (15 points) For each of the following matrices A, compute bases for $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$.

(a) $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ (c) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix}$

(a) The matrix is already in reduced row echelon form. The basis for the column space is the three pivot columns:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The null space is given by the standard algorithm (the fourth column is the only free column):

$$\left\{ \begin{pmatrix} -1 \\ -2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

The row space is given by the there non-zero rows:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

(b) We first row reduce to get

$$A \sim \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the column space's basis is the first two columns of the original matrix, the null space's basis has one vector, and the row space's basis is the first two rows of the row-reduced matrix:

$$\text{basis of } \operatorname{Col} A = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\text{basis of } \operatorname{Nul} A = \left\{ \begin{pmatrix} -\frac{1}{4} \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\text{basis of } \operatorname{Row} A = \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

(c) Row reduction:

$$A \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So, a basis for $\operatorname{Col} A$ is the three columns of A, the basis for $\operatorname{Nul} A$ is the empty set $\{\}$ (also written $\emptyset$), and a basis for $\operatorname{Row} A$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Make sure to use the nonzero rows of the row echelon form! Otherwise you would mistakenly write $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \right\}$ as a basis, which is a dependent set.

To compute a basis for $\operatorname{Row} A$, it is also OK to compute a basis for $\operatorname{Col} A^T$, since $\operatorname{Row} A = \operatorname{Col} A^T$. This method will give different bases from those given above.

2. (15 points) $\mathbb{R}^{m\times n}$ is the vector space of $m \times n$ matrices with the usual addition and scalar multiplication.

(a) Let W be the set of all 2×2 matrices A where $A\vec{x} = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ has at least one solution. Determine whether or not W is a subspace of $\mathbb{R}^{2\times 2}$, and if it is, give its dimension.

The 2×2 zero matrix $A = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ is a matrix where $A\vec{x} = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ has no solutions. This means W is missing the zero vector, so it is not a subspace of $\mathbb{R}^{2\times 2}$.

Note: having at least one solution to $A\vec{x} = \vec{b}$ doesn't imply the columns of A span $\mathbb{R}^2$! This was a common invalid deduction. All having a solution means is that $\vec{b}$ in $\operatorname{Col} A$ (and since $\operatorname{Col} A$ is a subspace, that $\operatorname{Span}\{\vec{b}\} \subset \operatorname{Col} A$). The zero matrix definitely doesn't have any nonzero $\vec{b}$ in its column space.

(b) Let U be the set of all 3×3 matrices A satisfying $A^T = -A$. Determine whether or not U is a subspace of $\mathbb{R}^{3\times 3}$, and if it is, give its dimension.

The first way we could show it is a subspace:

(1) Let A, B be two matrices in U. We can see $A+B$ is in U as well since $(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B)$, so $A+B$ also has the required property.

(2) Let A be a matrix in U and $c \in \mathbb{R}$. We can see cA is in U as well since $(cA)^T = cA^T = c(-A) = -(cA)$, so cA also has the required property.

Since both properties are proved, U is a subspace of $\mathbb{R}^{3\times 3}$. (No need to check $0 \in U$ unless disproving.)

The second way is to notice that $T(A) = A^T + A$ is a linear transformation. This is because $T(A+B) = (A+B)^T + (A+B) = (A^T + A) + (B^T + B) = T(A) + T(B)$ and $T(cA) = (cA)^T + cA = c(A^T + A) = cT(A)$. Then, $U = \ker T$, and kernels are subspaces.

For dimension, a general matrix A can be written as

$$A = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$$

so $A^T = -A$ implies nine equations (many superfluous). The essential ones:

$$\begin{aligned} a &= -a\\ e &= -e\\ i &= -i\\ b &= -d\\ g &= -c\\ h &= -f \end{aligned}$$

and this system has three free variables, b, c, and f, with $a = e = i = 0$. So, U is three-dimensional.

3. (15 points) Let $A = \left(\begin{smallmatrix}5 & -6 & 0\\ 3 & -4 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$. Its characteristic polynomial is $p_A(\lambda) = (1+\lambda)(1-\lambda)(2-\lambda)$, which you may use if you demonstrate how to compute it.

(a) Find all $c \in \mathbb{R}$ so that the matrix $A - c^2 I_3$ is **not** invertible.

First, let's calculate the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I_3) &= \det\begin{pmatrix} 5-\lambda & -6 & 0\\ 3 & -4-\lambda & 0\\ 0 & 0 & 1-\lambda\end{pmatrix}\\ &= (1-\lambda)\det\begin{pmatrix} 5-\lambda & -6\\ 3 & -4-\lambda\end{pmatrix}\\ &= (1-\lambda)((5-\lambda)(-4-\lambda)+18)\\ &= (1-\lambda)(\lambda^2 - \lambda - 2) = (1-\lambda)(1+\lambda)(2-\lambda) \end{aligned}$$

The matrix $A - c^2 I_3$ is not invertible when $\det(A - c^2 I_3) = 0$. This is exactly when c^2 is an eigenvalue, since $\det(A - \lambda I_3) = 0$ if and only if λ is an eigenvalue. Thus, $c^2 = 1, -1, 2$, which implies $c = \pm 1, \pm\sqrt{2}$ (we've discarded the roots of -1 since c is presumed to be real, since the problem asks for $c \in \mathbb{R}$).

(b) Compute $A^{22}\begin{pmatrix}1\\1\\1\end{pmatrix}$ by first diagonalizing A.

The eigenvalues of A are $1, -1, 2$, from the roots of the characteristic polynomial. Let us determine bases for the eigenspaces.

For $\lambda = 1$,

$$\mathrm{Nul}(A - I_3) = \mathrm{Nul}\begin{pmatrix}4 & -7 & 0\\ 3 & -5 & 0\\ 0 & 0 & 0\end{pmatrix}$$

We could go on and row reduce to show that the nullspace is only one-dimensional, or use the fact that there are two other eigenspaces, so it *must* be one-dimensional. We can see that $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is an eigenvector since it is in this nullspace.

For $\lambda = -1$,

$$\mathrm{Nul}(A + I_3) = \mathrm{Nul}\begin{pmatrix}6 & -6 & 0\\ 3 & -3 & 0\\ 0 & 0 & 2\end{pmatrix} = \mathrm{Nul}\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

so $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is an eigenvector.

For $\lambda = 2$,

$$\mathrm{Nul}(A - 2I_3) = \mathrm{Nul}\begin{pmatrix}3 & -6 & 0\\ 3 & -6 & 0\\ 0 & 0 & -1\end{pmatrix} = \mathrm{Nul}\begin{pmatrix}1 & -2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

so $\begin{pmatrix}2\\1\\0\end{pmatrix}$ is an eigenvector.

Hence, we obtain the following diagonalization $A = PDP^{-1}$:

$$D = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix} \qquad P = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 1\\ 1 & 0 & 0\end{pmatrix}$$

Since $A^{22} = PD^{22}P^{-1}$, then to calculate $A^{22}\begin{pmatrix}1\\1\\1\end{pmatrix}$ we may calculate $PD^{22}P^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$. The vector $P^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$ is the coordinate vector of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ relative to basis P (that is, the solution to $P\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$), and this is obviously[a] $\begin{pmatrix}1\\1\\0\end{pmatrix}$. Then, $D^{22}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1^{22}\\(-1)^{22}\\0\end{pmatrix}$, so then we calculate $P\begin{pmatrix}1\\1\\0\end{pmatrix}$, which is $\begin{pmatrix}1\\1\\1\end{pmatrix}$, which is the answer.

[a] I mean this strictly in the sense that you can see it's true by looking at it.

4. (15 points) Let W be the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

(a) Compute $\operatorname{proj}_W \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

We first construct an orthogonal basis. The first vector $\vec{u}_1$ is the first vector, but the second vector $\vec{u}_2$ is (by the Gram-Schmidt process)

$$\vec{u}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Then, we compute

$$\operatorname{proj}_W \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Alternatively, we compute $\vec{x} - \operatorname{proj}_{W^\perp} \vec{x}$. To find a basis for $W^\perp$, we find a basis for $\operatorname{Nul} \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \end{pmatrix} = \operatorname{Nul} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \operatorname{Nul} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. Then,

$$\operatorname{proj}_{W^\perp} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

So, $\operatorname{proj}_W \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

Even another way is to take the matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & -3 \\ 1 & 4 \end{pmatrix}$, find the least squares solution $A\hat{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ by solving $A^T A\hat{x} = A^T \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, and then computing $A\hat{x}$, which is the projection. $A^T A = \begin{pmatrix} 3 & 9 \\ 9 & 29 \end{pmatrix}$ and $A^T \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix}$. Then, solving this system, $\hat{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, so $\operatorname{proj}_W \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = A\hat{x} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

(b) Compute the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = \operatorname{proj}_W \vec{x}$.

Two ways to do this. The first is to compute $\begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{pmatrix}$ using the orthogonal basis $\{\vec{u}_1, \vec{u}_2\}$ from (a).

$$\begin{aligned} T(\vec{e}_1) &= \frac{\vec{u}_1 \cdot \vec{e}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{u}_2 \cdot \vec{e}_1}{\vec{u}_2 \cdot \vec{u}_2} = \frac{1}{3}\vec{u}_1 - \frac{1}{2}\vec{u}_2 = \begin{pmatrix} 5/6 \\ -1/3 \\ -1/6 \end{pmatrix} \\ T(\vec{e}_2) &= \frac{\vec{u}_1 \cdot \vec{e}_2}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{u}_2 \cdot \vec{e}_2}{\vec{u}_2 \cdot \vec{u}_2} = -\frac{1}{3}\vec{u}_1 + 0\vec{u}_2 = \begin{pmatrix} -1/3 \\ 1/3 \\ -1/3 \end{pmatrix} \\ T(\vec{e}_3) &= \frac{\vec{u}_1 \cdot \vec{e}_3}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{u}_2 \cdot \vec{e}_3}{\vec{u}_2 \cdot \vec{u}_2} = \frac{1}{3}\vec{u}_1 + \frac{1}{2}\vec{u}_2 = \begin{pmatrix} -1/6 \\ -1/3 \\ 5/6 \end{pmatrix} \end{aligned}$$

Thus, the matrix of T is $[T] = \begin{pmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{pmatrix}$

The second is to make an orthonormal basis and compute UU^T. We have $U = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$, and then UU^T is the same matrix as the one above.

(c) Compute the dimension of $W^\perp$.

Dimensions of a subspace and its orthogononal complement satisfy $\dim W + \dim W^\perp = \dim \mathbb{R}^3$, so $\dim W^\perp = 3 - 2 = 1$. The nullspace calculation for one of the answers to (a) also gives 1 as the dimension.

5. (15 points) $\mathbb{P}_2$ is the vector space of polynomials whose degree is at most 2. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $T(p(x)) = p(x) - p(-x)$ for $p(x) \in \mathbb{P}_2$. For instance, $T(x+1) = (x+1) - (-x+1) = 2x$.

(a) Show that T is a linear transformation.

The way we show something is a linear transformation is to check the two properties from the definition:

(1) For $p(t), q(t)$ polynomials in $\mathbb{P}_2$, we have $T(p(t) + q(t)) = (p(x) + q(x)) - (p(-x) + q(-x)) = (p(x) - p(-x)) + (q(x) - q(-x)) = T(p(x)) + T(q(x))$.

(2) For $p(t)$ a polynomial in $\mathbb{P}_2$ and $c \in \mathbb{R}$, we have $T(cp(x)) = (cp(x)) - (cp(-x)) = c(p(x) - p(-x)) = cT(p(x))$.

Therefore, T is a linear transformation.

(No need for $T(0) = 0$. This is implied by (2).)

(b) Compute bases for the kernel of T and for the image of T. (*Range* is a synonym for *image*.)

We have to go to coordinates, essentially, to answer this. We calculate $T(p(x))$ where $p(x) = ax^2+bx+c$. Then, $T(ax^2 + bx + c) = (ax^2 + bx + c) - (a(-x)^2 + b(-x) + c) = 2bx$.

The kernel is $\ker T = \{p(x) \in \mathbb{P}_2 : T(p(x)) = 0\} = \{ax^2 + bx + c : a, b, c \in \mathbb{R} \text{ and } T(ax^2 + bx + c) = 0\} = \{ax^2 + bx + c : a, b, c \in \mathbb{R} \text{ and } 2bx = 0\}$. Since $2bx$ is the zero polynomial if and only if $b = 0$, then $\ker T = \{ax^2 + c : a, c \in \mathbb{R}\} = \text{Span}\{x^2, 1\}$. Since x^2 and 1 are linearly independent polynomials, $\{x^2, 1\}$ is a basis for $\ker T$.

The image is the set of all images, which we calculated to be $\{2bx : b \in \mathbb{R}\}$. This is $\text{Span}\{x\}$, so $\{x\}$ is a basis for $\operatorname{im} T$.

(c) Show that T is neither one-to-one nor onto.

Not one-to-one: (any of these will do)

- $\dim \ker T > 0$.
- $T(x) = T(x + 1)$ (example of two things mapping to the same thing)
- $T(1) = 0$ (example of a nonzero thing in the kernel)

Not onto: (any of these will do)

- $\dim \operatorname{im} T < \dim \mathbb{P}_2$ (that is, $1 < 3$)
- $T(p(x)) = 1$ has no solutions since $T(p(x))$ is always of the form $2bx$ for some b.