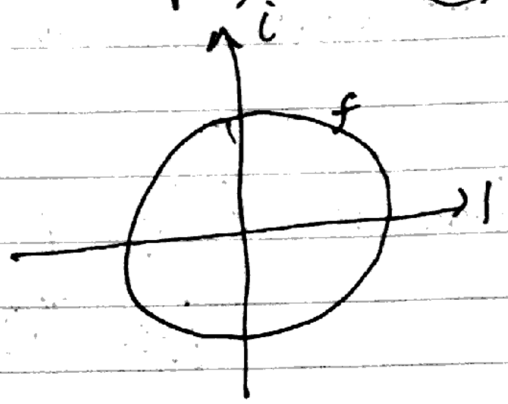
1 Aug 10

Fourier series ~~a~~ a good way

I find sin/cos somewhat arbitrary without some complex ($\mathbb{C}$) fundamentals.

Setup: you have a function defined on the complex unit circle and you want to approximate it with polynomials. (and conjugates of polynomials) → i.e. "Laurent series" instead of "Taylor series"



for each θ, $f(e^{i\theta})$ is some number

This is ~~by~~ a geometric realization of periodic functions: $g(\theta) = f(e^{i\theta})$ is periodic with period 2π, and any periodic function can be wrapped around a circle.

Key idea: for $z = e^{i\theta}$, $z^n = e^{ni\theta} = \cos(n\theta) + i\sin(n\theta)$

and $\bar{z}^n = (e^{-i\theta})^n = e^{-ni\theta} = \cos(n\theta) - i\sin(n\theta)$

Want to find coefficients $\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots$ such that $f(z) \approx \sum_{n=-\infty}^{\infty} c_n z^n$ for $z = e^{i\theta}$

(this can be thought of as a polynomial interpolation problem. DFT is exactly this)

Using Euler's identity

$$f(z) \approx \sum_{n=-\infty}^{\infty} c_n(\cos(n\theta) + i\sin(n\theta))$$

$$\approx c_0 + \sum_{n=1}^{\infty}\left((c_n+c_{-n})\cos(n\theta) + (c_n-c_{-n})i\sin(n\theta)\right)$$

(so can solve for book's coefficients)

Hermitian inner product:

$$\langle f,g\rangle = \int_0^{2\pi} \overline{f(e^{i\theta})}\, g(e^{i\theta})\, d\theta$$

properties: $\langle f,g\rangle = \overline{\langle g,f\rangle}$

... some others, easily calculated

$\langle f,f\rangle \in \mathbb{R}$ and $\langle f,f\rangle \geq 0$

$= 0$ iff $f=0$

Claim: if $m \neq n$, z^m and z^n orthogonal.

$$\langle z^m, z^n\rangle = \int_0^{2\pi} \overline{e^{mi\theta}}\, e^{ni\theta}\, d\theta$$

$$= \int_0^{2\pi} e^{(n-m)i\theta}\, d\theta$$

$$= \left[\frac{1}{(n-m)i} e^{(n-m)i\theta}\right]_0^{2\pi} = \frac{1}{(n-m)i}(1-1) = 0$$

$$\langle z^n, z^n\rangle = \int_0^{2\pi} \overline{e^{ni\theta}}\, e^{ni\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

so set of all $\frac{1}{2\pi} z^n$ $(n \in \mathbb{Z})$ is orthonormal set.

Interesting facts:

If $f = \sum_{n=-\infty}^{\infty} c_n z^n \qquad g = \sum_{n=-\infty}^{\infty} d_n z^n$

$$\langle f, g\rangle = \langle \sum_n c_n z^n, \sum_m d_m z^m \rangle$$

$$= \sum_n \overline{c_n} \langle z^n, d_m z^m \rangle$$

$$= \sum_n \sum_m \overline{c_n} d_m \underbrace{\langle z^n, z^m \rangle}_{\text{zero if } n \neq m}$$

$$= \sum_n \overline{c_n} d_n \cdot 2\pi = 2\pi \sum_n \overline{c_n} d_n$$

$$= 2\pi\, \vec{c}^{\,*} d_n$$

conjugate transpose

So, $\langle f, f\rangle = 2\pi \sum_n \overline{c_n} c_n$

$$= 2\pi \sum_n \|c_n\|^2$$

(Parseval's theorem)
this is "energy" in wave

$\langle z^n, f\rangle = 2\pi c_n$, so $c_n = \dfrac{\langle z^n, f\rangle}{2\pi}$

Fourier transform: Given f piecewise continuous on unit circle: $c_n = \dfrac{\langle z^n, f\rangle}{2\pi}$.

$\displaystyle\sum_{n=-\infty}^{\infty} \frac{\langle z^n, f\rangle}{2\pi} z^n$ is Fourier series for f.

This is a <u>formal</u> series. It might not converge!

<u>Thm</u> Pointwise convergence. If f, f' are piecewise continuous, and at $e^{i\theta}$, v^- and v^+ are one-sided limits

$$\sum_{n=-\infty}^{\infty} c_n z^n \text{ converges to } \frac{v^- + v^+}{2}$$

<u>ex</u> Let $f(e^{i\theta}) = \begin{cases} 1 & \text{if } 0 \leq \theta < \pi \\ 0 & \text{if } \pi \leq \theta < 2\pi \end{cases}$

$$\langle z^n, f \rangle = \int_0^{2\pi} \overline{e^{ni\theta}} f(e^{i\theta})\, d\theta$$

$$= \int_0^{\pi} e^{-ni\theta}\, d\theta = \left[\frac{-1}{ni} e^{ni\theta}\right]_0^{\pi} \quad \text{if } n \neq 0$$

$\rightarrow$ if $n = 0$, $= \pi$

$$= \frac{-1}{ni} e^{\pi i n} + \frac{1}{ni}$$

n	$e^{\pi i n}$
0	1
1	-1
2	1
3	-1

since $(e^{\pi i} = -1)$

$$= \frac{-1}{ni}(-1)^n + \frac{1}{ni}$$

$$= \frac{1 - (-1)^n}{ni}$$

so: if n even, $= 0$

if n odd $= \frac{2}{ni}$

$$f(z) \approx \frac{\pi}{2\pi} + \sum_{n \text{ odd}} \frac{2}{2\pi n i} z^n = \frac{1}{2} + \sum_{n \text{ odd}} \frac{1}{n\pi i} z^n$$

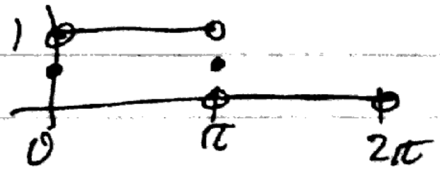
Going to sin/cos

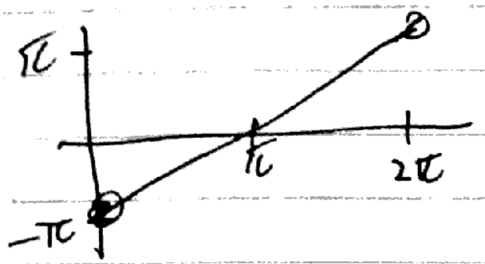
$$\approx \frac{1}{2} + \sum_{n \text{ odd}} \frac{1}{n\pi} \sin(n\theta)$$

Discontinuity at $\theta = 0$ and $\theta = \pi$

~~Let's see how~~ $\theta = 0 : \frac{1}{2}$

$\theta = \pi : \frac{1}{2}$





$f(e^{i\theta}) = \theta - \pi$

$$\langle z^n, f(z)\rangle = \int_0^{2\pi} \overline{e^{ni\theta}}\, f(e^{i\theta})\, d\theta$$

$$= \int_0^{2\pi} e^{-ni\theta}(\theta - \pi)\, d\theta$$

$$= \int_0^{2\pi} \theta e^{-ni\theta}\, d\theta - \pi \int_0^{2\pi} e^{-ni\theta}\, d\theta$$

if $n=0$, $$= \int_0^{2\pi} \theta\, d\theta - \pi \int_0^{2\pi} d\theta$$

$$= \left[\tfrac{1}{2}\theta^2\right]_0^{2\pi} - \pi\left[1\right]_0^{2\pi}$$

$$= \tfrac{1}{2}(4\pi^2) - \pi(2\pi) = 0.$$

if $n \neq 0$ $\quad u = \theta \quad dv = e^{-ni\theta}d\theta$

$du = d\theta \quad v = \frac{-1}{ni} e^{-ni\theta}$

$$= \left[\frac{-\theta}{ni} e^{-ni\theta}\right]_0^{2\pi} - \pi\left[\frac{-1}{ni} e^{-ni\theta}\right]_0^{2\pi}$$

$$= \frac{-2\pi}{ni} e^{-2\pi ni} - \pi\left(\frac{-1}{ni} e^{-2\pi i n} + \frac{1}{ni}\right)$$

$$= \frac{-\pi}{ni} - \frac{\pi}{ni} = -\frac{2\pi}{ni}$$

so $c_n = \dfrac{-2\pi/ni}{2\pi} = \dfrac{-1}{ni}$

$$f(z) \approx \sum_{n\neq 0} \frac{-1}{ni} z^n = \sum_{n=1}^{\infty} \frac{-1}{n}\sin(n\theta)$$