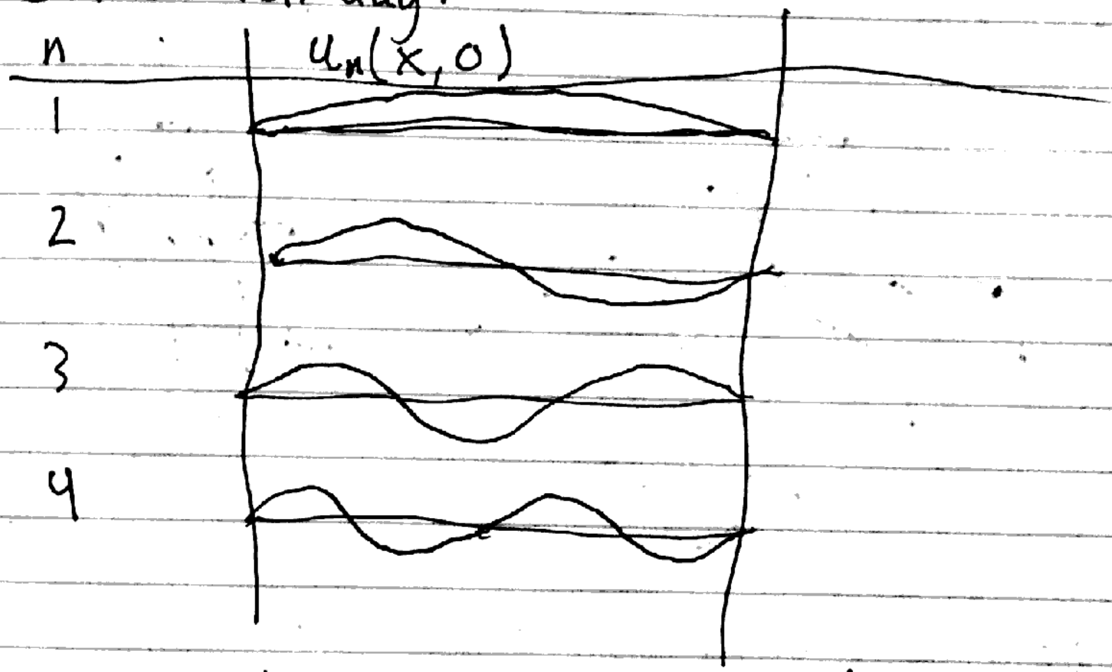
Aug 9

Yesterday, we found solutions to the heat equation $u_t = \beta u_{xx}$ satisfying the boundary conditions $u(0,t) = u(L,t) = 0$ of the form

$$u_n(x,t) = e^{-\beta(n\pi/L)^2 t} \sin(n\pi x/L)$$

for $n = 1, 2, 3, \ldots,$ using separation of variables.

Let's examine these closer. For $t=0$, we have the following:

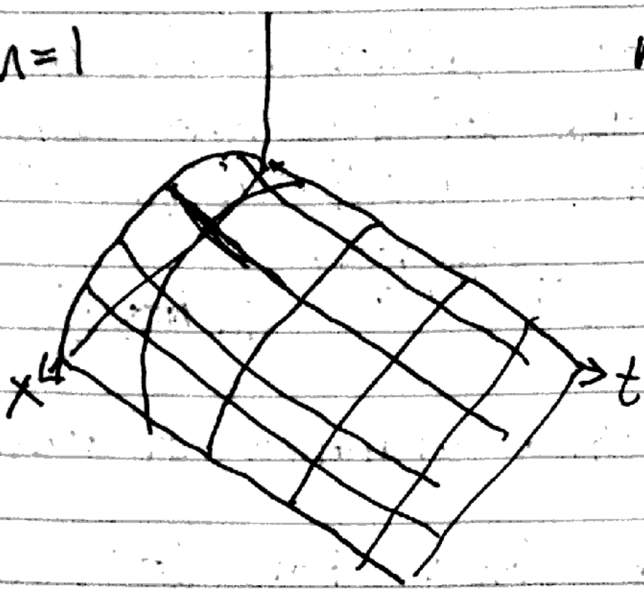


They are all waves which complete an integer number of half periods of sin across $[0, L]$.
What about the $\exp(-\beta(n\pi/L)^2 t)$ part?
For larger n, $-\beta(n\pi/L)^2$ is more negative, so the corresponding u_n goes to zero faster.

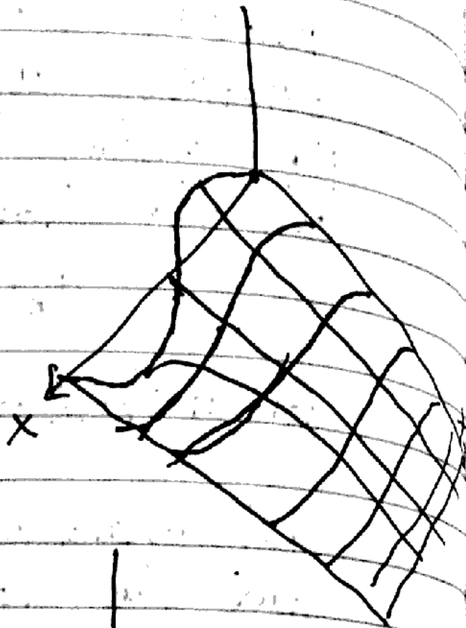
Let's look at time and space simultaneously.

2

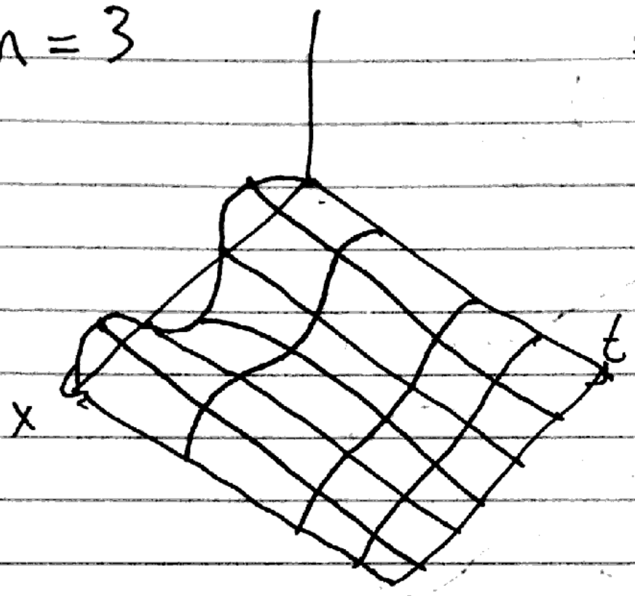
n=1



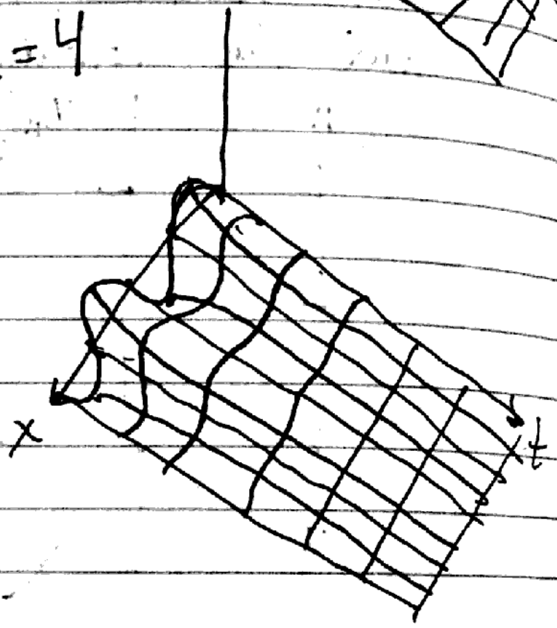
n=2



n=3



n=4



Linear combinations of these graphs are graphs of solutions to the heat equation with the given boundary conditions. Separability represents that we have solutions where the two sets of orthogonal curves in each diagram can be independently calculated, and that curves going in the same direction are just scalings of each other.

Whether these are a "basis" is left to Fourier series.

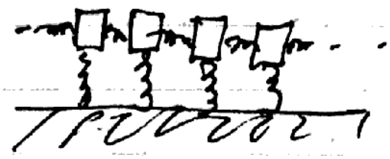
The Wave equation

Let's ponder the following equation into existence for a vibrating string:

$$u_{tt} = \alpha^2 u_{xx}$$

$\uparrow$ acceleration (force) $\qquad$ $\uparrow$ concavity

We could derive this as a limit of oscillators connected in sequence by springs



But we won't! Let's consider the boundary value problem $u(0,t) = u(L,t) = 0$

Separation of variables works here, too. Suppose a solution is of the form $u(x,t) = X(x)T(t)$ (i.e., $X(x)$ is merely scaled through time).

$$u_{tt}(x,t) = X(x)T''(t) \qquad u_{xx}(x,t) = X''(x)T(t)$$

substituting these:

$$X(x)T''(t) = \alpha^2 X''(x)T(t)$$

$$\frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

Again, t, x independently vary, so each ratio is constant. Say $-\lambda$ (since will make math nicer)

get: $X''(x) + \lambda X(x) = 0$ and $T''(t) + \alpha^2 \lambda T(t) = 0$

The boundary conditions (assuming $T(t)$ not always zero) mean $X(0)=X(L)=0$.

Solutions: $r^2+\lambda=0 \Rightarrow r=\pm\sqrt{\lambda}\,i$

we already saw that $\lambda>0$ since otherwise we cannot meet the boundary conditions without X being constant 0. For $\lambda>0$,

$$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$0=X(0) = c_1\cdot 1 + c_2\cdot 0$$
$$0=X(L) = c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L)$$

$c_1=0$, c_2 anything so long as $\sqrt{\lambda}L$ multiple of π

$$\sqrt{\lambda} = \frac{\pi n}{L} \qquad n=1,2,3,\dots$$

$$X_n(x) = \sin\frac{n\pi x}{L} \quad \text{with } \lambda=\left(\frac{n\pi}{L}\right)^2$$

for this λ, we have

$$T''(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 T(t) = 0$$

so $T_n(t) = c_n\cos\left(\frac{n\pi\alpha}{L}t\right) + d_n\sin\left(\frac{n\pi\alpha}{L}t\right)$

which gives

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n\cos\left(\frac{n\pi\alpha}{L}t\right) + d_n\sin\left(\frac{n\pi\alpha}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

This ∞ sum technically means only finitely many c_n, d_n are nonzero for these to be solutions, but we will see that, when this formal sum converges, it is a solution. This isn't actually a basis since convergence involves limits! (It's a "basis" in that every ~~continuous~~ differentiable function is a limit of

To satisfy an init. boundary condition,

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} d_n \frac{n\pi\alpha}{L} \sin\left(\frac{n\pi x}{L}\right)$$

so position and velocity of points of string.
Given f, g, want to satisfy

$$u(x,0) = f(x) \qquad 0 \le x \le L$$
$$u_t(x,0) = g(x) \qquad 0 \le x \le L$$

for boundary condition. This again involves Fourier series.

Let's understand what is going on with the solutions we have obtained. Through time, at wherever $\sin\left(\frac{n\pi x}{L}\right)$ is 1, we see the string travels along $\sum_{n=1}^{\infty}\left(c_n \cos\left(\frac{n\pi\alpha}{L}t\right) + d_n \sin\left(\frac{n\pi\alpha}{L}t\right)\right)$

$$= \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi\alpha}{L}t + b_n\right)$$

frequency : $\frac{n\pi\alpha/L}{2\pi} = \frac{n\alpha}{2L}$ times per second.

$\frac{\alpha}{2L}, \quad \frac{2\alpha}{2L}, \quad \frac{3\alpha}{2L}, \quad \frac{4\alpha}{2L}, \quad \ldots$

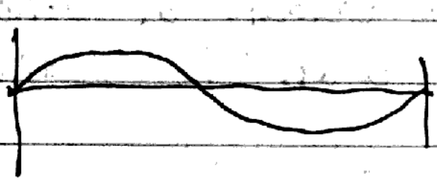
fundamental "overtones" . . .

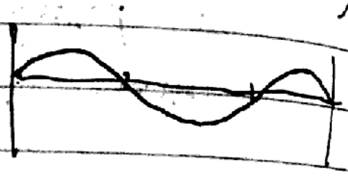
For a particular frequency $\frac{n\alpha}{2L}$, the corresponding standing wave on the string has n nodes.

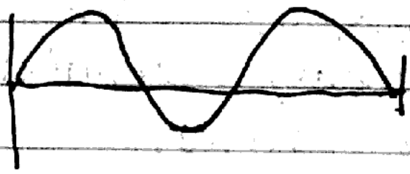
(This suggests that driving, say by shaking, the string at a particular frequency will cause a particular standing wave pattern)

ex Init condition $\cos\left(\frac{2\pi\alpha}{L}t\right) + \cos\left(\frac{3\pi\alpha}{L}t\right)$

solution: $\cos\left(\frac{2\pi\alpha}{L}t\right)\sin\left(\frac{2\pi x}{L}\right) + \cos\left(\frac{3\pi\alpha}{L}t\right)\sin\left(\frac{3\pi x}{L}\right)$







Fourier series

A *periodic* function is where $f(x+L) = f(x)$. L is a *period*. The *fundamental period* is the minimal period.

An *even* function is one where $f(x) = f(-x)$.

An *odd* function is one where $f(x) = -f(-x)$.