Aug 8

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Partial differential equations

So far we have dealt with functions of one variable, and equations involving their derivatives. However, many important physical systems involve multiple variables, like time and position. We need some way of describing rate of change in this new situation.

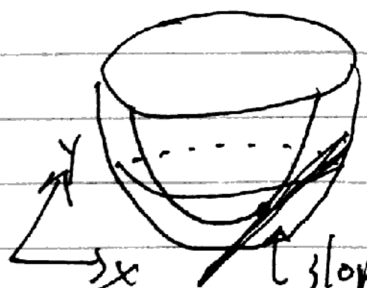
Suppose $u(x,t)$ is a real-valued function of x and t. The partial derivative of u with respect to x is

$$\frac{\partial u}{\partial x}(x,t) = \lim_{h\to 0} \frac{u(x+h,t) - u(x,t)}{h}$$

also written $u_x(x,t)$. This is the derivative holding every other variable constant.

ex Let $f(x,y) = x^2 + 2y^2 + 1 + y$

$$f_x(x,y) = 2x$$
$$f_y(x,y) = 4y + 1$$



slope is $f_x(x,y)$ since tangent is in x-direction

The gradient is a row vector of all first partial derivs

On the homework last week, there was a problem concerning water flow in an ice tray. With u_i the height of water in cell i, we had $u_i' = (u_{i+1} - u_i) + (u_{i-1} - u_i) = u_{i-1} - 2u_i + u_{i+1}$
Let's take a limit on the # of cells, n.
Instead of cell i, let's say the cell is located at $x = \frac{i}{n}$, and let $u(\frac{i}{n}, t)$ be the amount of water in cell i at time t.

$$\frac{\partial}{\partial t} u\left(\tfrac{i}{n}, t\right) = \frac{1}{n^2}\left(u\left(\tfrac{i-1}{n}, t\right) - 2u\left(\tfrac{i}{n}, t\right) + u\left(\tfrac{i+1}{n}, t\right)\right)$$

↑ scales diff. of differences properly.

let $x = \frac{i}{n}$ be fixed, so as $n \to \infty$, $i \to \infty$ to maintain the ratio.

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{n^2}\left(u\left(x - \tfrac{1}{n}, t\right) - 2u(x,t) + u\left(x + \tfrac{1}{n}, t\right)\right)$$

$$= \frac{1}{n}\left(\frac{u(x+\frac{1}{n}, t) - u(x,t)}{n} - \frac{u(x,t) - u(x-\frac{1}{n}, t)}{n}\right)$$

$$\text{(as } n \to \infty\text{)} \quad \approx \frac{1}{n}\left(\frac{\partial u}{\partial x}(x,t) - \frac{\partial u}{\partial x}\left(x - \tfrac{1}{n}, t\right)\right)$$

$$\approx \frac{\partial^2 u}{\partial x^2}(x,t)$$

Thus, the continuous case is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

For a constant β, we get

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

the homogeneous one-dimensional <u>heat equation</u>
(water in cells is like "caloric fluid")

This is also written

$$u_t = \beta u_{xx} \quad \text{or} \quad u_t = \beta \Delta u$$

$\Delta u := u_{xx} + u_{yy} + \cdots$ is the Laplacian, also written $\nabla^2 u$. Steady-state is $\Delta u = 0$.

When there is a heat/fluid source, the equation is

$$u_t = \beta \Delta u + P$$

where $P(x,t)$ gives the rate of new heat entering at pos x at time t.

Instead of ~~boundary~~ initial conditions, we get to think about boundary conditions ~~for~~ boundary condition problems. For the heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial t^2}(x,t) \qquad 0<x<L,\ t>0$$

$$u(0,t) = u(L,t) = 0, \quad t>0$$

$$u(x,0) = f(x), \quad 0<x<L$$

is ~~common~~ representative. A length-L object whose ends are fixed at 0°C, with initial heat f along wire.

In ~~general~~ practice, finding analytic solns is impractical, so there are many numerical methods. We can solve this boundary value problem using separation of variables, though. This converts the problem into single-var problems.

Guess: we can solve $u(x,t) = X(x)T(t)$ for X and T.

$$u_t(x,t) = X(x)T'(t)$$
$$u_x(x,t) = X'(x)T(t) \qquad u_{xx}(x,t) = X''(x)T(t)$$

Substituting,

$$X(x)T'(t) = \beta X''(x)T(t)$$

so $$\frac{T'(x)}{\beta T(x)} = \frac{X''(t)}{X(t)}$$

Fact: holding x or t constant and letting the other vary implies both ratios are constant. Let $-\lambda$ be this constant:

$$\frac{T'(x)}{\beta T(x)} = -\lambda \qquad \frac{X''(t)}{X(t)} = -\lambda$$

so now to solve $T' + \lambda\beta T(x) = 0$
and $X''(t) + \lambda X(t) = 0$.

The boundary conditions $u(0,t) = u(L,t) = 0$ imply $X(0)T(t) = X(L)T(t) = 0$.
Either $T(t) = 0$ for all t (and get zero solution) or $X(0) = X(L) = 0$.

Let's solve

$$X''(x) + \lambda X(x) = 0 \qquad X(0) = X(L) = 0.$$

$$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$$

<u>Case I</u> $\lambda < 0$. Then roots real.

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$0 = X(0) = c_1 + c_2$$

$$0 = X(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$$

$c_1 = -c_2$ by 1^{st}

so $c_1 \underbrace{(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L})}_{\text{not zero}} = 0$

so $c_1 = 0$, so $c_2 = 0$.

~~No nontrivial~~ Thus X is trivial solution in this case. Nothing new here.

<u>Case II</u> $\lambda = 0$. Repeated root,

$$X(x) = c_1 + c_2 x$$

$$\left.\begin{aligned} 0 &= X(0) = c_1 \\ 0 &= X(L) = c_1 + c_2 L \end{aligned}\right\} \rightarrow c_1 = c_2 = 0.$$

Same situation

Case III $\lambda > 0$. Two imaginary roots

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$0 = X(0) = c_1 + 0 \Rightarrow c_1 = 0$$

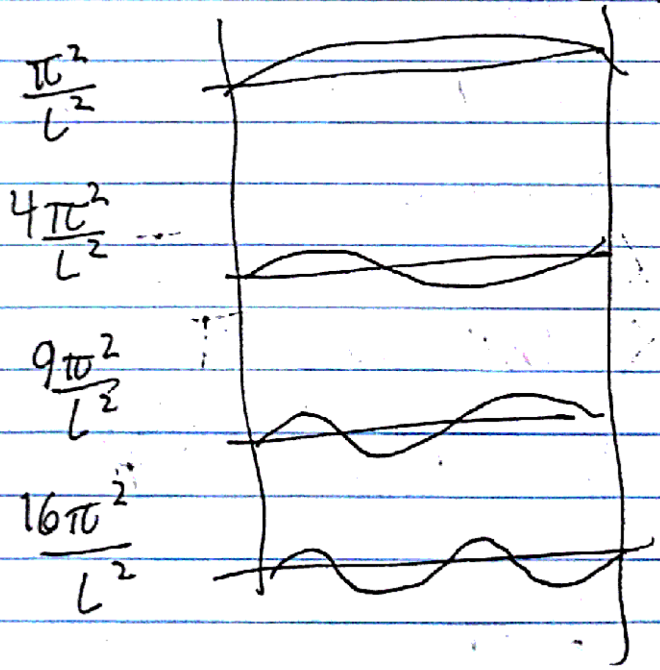
$$0 = X(L) = c_2 \sin \sqrt{\lambda} L$$

$$\Rightarrow \sqrt{\lambda} L = \pi n \text{ for some } n = 1, 2, 3, \ldots$$

$$\text{so } \sqrt{\lambda} = \frac{\pi n}{L}$$

$$\text{thus, } X(x) = c_2 \sin\left(\frac{\pi n x}{L}\right)$$

The eigenfunctions are $X_n(x) = \sin\left(\frac{\pi n x}{L}\right)$ with eigenvalue $\lambda = \left(\frac{\pi n}{L}\right)^2$.



These are actually orthogonal according to $\langle f, g \rangle = \int_0^L f(t) g(t)\, dt$.

Now for $T' + \beta\lambda T = 0$.
For $\lambda = \left(\frac{n\pi}{L}\right)^2$ ($n = 1, 2, 3, \cdots$)

$$T'(t) + \beta\left(\frac{n\pi}{L}\right)^2 T(t) = 0.$$

$$T_n(t) = e^{-\beta(n\pi/L)^2 t}$$ is corresp. solution

$$u_n(x,t) = X_n(x)\,T_n(t) = e^{-\beta(n\pi/L)^2 t}\sin(n\pi x/L)$$

These are lin. indep. solutions.

ex $\quad u_t = 7u_{xx} \qquad 0<x<\pi \qquad t>0$

$u(0,t) = u(\pi,t) = 0 \qquad t>0$

$u(x,0) = 3\sin(2x) - 6\sin(5x)$

$$u(x,0) = \sum_n c_n u_n(x,t) = \sum_n c_n e^0 \sin(n\pi x/\pi)$$

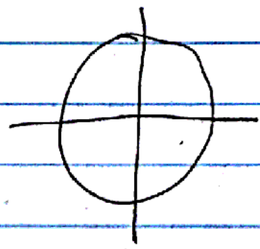
$c_2 = 3$
$c_5 = -6$

$$u(x,t) = 3e^{-7(2)^2 t}\sin(2x) - 6e^{-7(5)^2 t}\sin(5x)$$ is solution.

This isn't exactly a basis. Though, because of Fourier series, every function is "almost" an infinite lin. comb. of $\sin(n\pi x/L)$, so this is "complete" in a limit sense.

Strings next.

$u_{xx} + u_{yy} = 0$



Say $f(x,y)$ defined on circle $x^2+y^2=1$

$u(x,y) = f(x,y)$ for such points.

$u(x,y) = X(x)\,Y(y)$

$X''(x)\,Y(y) = -X(x)\,Y''(y)$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \lambda$$

boundary condition: $X(x)\,Y(y) = f(x,y)$ if $x^2+y^2=1$

This seems difficult with this bndry cond.

$u(x,y) = f(\sqrt{x^2+y^2})\,g(\arg(x+iy))$

guess: $u(x,y) = (x+iy)^n$ is solution

$u_x = n(x+iy)^{n-1}$ $\quad u_{xx} = n(n-1)(x+iy)^{n-2}$

$u_y = in(x+iy)^{n-1}$ $\quad u_{yy} = -n(n-1)(x+iy)^{n-2}$

good: $u_{xx}+u_{yy}=0$. $\bar{u}$ is also a solution.

~~u(x,y)~~ Can get real solutions out of these. Fourier series solves bndry cond