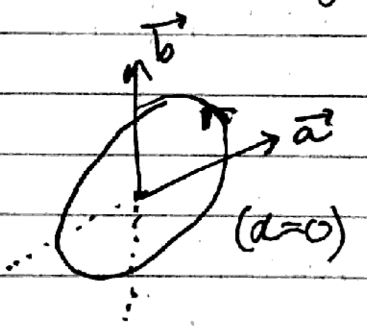
Aug 5

Yesterday, I had a question about knowing which way a solution travels around the origin for non-real eigenvalues. It turns out the answer was right in front of us!

Let $\lambda = \alpha - i\beta$ be an eigenvalue, with $\beta \geq 0$, and let $\vec{a} + i\vec{b}$ be the corresponding eigenvector. One of the two solutions in the fund. sol. set is

$$e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(-\beta t)\right)$$
$$= e^{\alpha t}\left(\vec{a}\cos(\beta t) + \vec{b}\sin(\beta t)\right)$$
$$= \underbrace{e^{\alpha t}}_{\text{scaling}}\underbrace{\begin{pmatrix}\vec{a} & \vec{b}\end{pmatrix}}_{2\times2 \text{ transformation matrix}}\underbrace{\begin{pmatrix}\cos\beta t \\ \sin\beta t\end{pmatrix}}_{\text{CCW circular path}}$$



So it seems

$\det\begin{pmatrix}\vec{a} & \vec{b}\end{pmatrix}$

being >0 means CCW

being <0 means CW

being $=0$ means real eigenvalue (no rotation)

One more thing about diagonalization:

the idea is to find bases which <u>uncouple</u> the system. Yesterday's example of two coupled mass-spring systems was diagonalizable, and it showed there were two uncoupled components ("modes")

1) of both moving in syncrony
2) of both moving in antisyncrony

The counter-intuitive thing here is that we believe masses are objects, indivisible individuals, but the addition of a spring between them means that certain motions of them together are the true, uncoupled "objects" of the system

(in quantum mechanics, particles are eigenvectors of some matrix. So the "particles" here are these two motions)

Matrix exponential

Recall: for diagonal D, $D^m = \begin{pmatrix} d_{11}^m & & 0 \\ & d_{22}^m & \\ 0 & \ddots & d_{nn}^m \end{pmatrix}$

$$\text{So, } \exp(D) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{pmatrix} d_{11}^m & & 0 \\ 0 & \ddots & d_{nn}^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{1}{m!} d_{11}^m & & 0 \\ & \ddots & \\ 0 & & \sum_{m=0}^{\infty} \frac{1}{m!} d_{nn}^m \end{pmatrix} = \begin{pmatrix} e^{d_{11}} & & 0 \\ & \ddots & \\ 0 & & e^{d_{nn}} \end{pmatrix}$$

"exponential of a diagonal matrix is diagonal of exponentials"

Property: $\exp\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^0 + \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}^1 + \frac{1}{2!} \cdots$

$$= I + 0 + 0 + \cdots$$

$$= I$$

"exponential of zero matrix is identity matrix"

Property: $\exp(A+B) = \exp(A)\exp(B)$ if $AB = BA$

(we actually showed this a few weeks ago! $AB=BA$ necessary for binomial theorem.)

consequence: $\exp(A(t+s)) = \exp(At)\exp(As)$
since $At\,As = As\,At$

consequence: $\exp(-A)\exp(A) = \exp(-A+A) = I$
since $-A\,A = A(-A)$

So: inverse of $\exp(A)$ is $\exp(-A)$
(that is, $\exp(A)^{-1} = \exp(-A)$)

property: $\exp(rI) = e^r I$ (since rI diagonal)

property: $\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots\right)$

$$= A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \cdots$$
$$= A\left(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots\right)$$
$$= A\exp(At)$$

So: $\exp(At)\vec{c}$ is a solution to $\vec{x}' = A\vec{x}$.

$\exp(At)$ is invertible, so has independent columns.
Thus: it is a fund. matrix of $\vec{x}' = A\vec{x}$!
($\exp(At)\vec{c}$ for varying $\vec{c}$ gives all solutions)

If $X(t)$ is some fundamental matrix, then
$$\exp(At) = X(t)X(0)^{-1}$$

Reason: Solution $\exp(At)\vec{c}$ has initial condition $\vec{x}(0) = \exp(A\cdot 0)\vec{c} = I\vec{c} = \vec{c}$.
Solution $\vec{y}(t) = X(t)X(0)^{-1}\vec{c}$ has init. cond.
$\vec{y}(0) = X(0)X(0)^{-1}\vec{c} = \vec{c}$. So $\vec{x} = \vec{y}$ by uniqueness thm.
Since $\vec{c}$ can equal $\vec{e}_1, \ldots, \vec{e}_n$, columns of

$\exp(At)$ and ~~exp~~ $X(t)X^{-1}(0)$ are the same.
(this suggests a way to compute exp: find a fund matrix!)

A *nilpotent* matrix A is one where $A^k = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ for some k.

ex $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

for $A^k = 0$,

$$\exp(At) = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{(k-1)!}A^{k-1}t^{k-1} + 0 + \cdots$$

so $\exp(At)$ is a finite calculation for nilpotent matrices

ex
$$\exp\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}t\right) = \exp\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

since $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + 0 + \cdots$$
$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\exp\begin{pmatrix} t & \\ & t \end{pmatrix} = \begin{pmatrix} e^t & \\ & e^t \end{pmatrix}$$

$$\text{so } \exp\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}t\right) = \begin{pmatrix} e^t & \\ & e^t \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

hence, $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\vec{x}$ has ↗ as fund matrix.

If λ is an eigenvalue of A with multiplicity m, then a <u>generalized eigenvector</u> is a nonzero element of $Nul((A-\lambda I)^m)$

That is, $\vec{u}$ is a generalized eigenvector if

$$(A-\lambda I)^m \vec{u} = \vec{0}.$$

Fact: $Nul((A-\lambda I)^m)$ is m-dimensional

This is a consequence of the Cayley-Hamilton theorem: if char poly is $(\lambda-r_1)^{m_1}(\lambda-r_2)^{m_2}\ldots$ then $(A-r_1I)^{m_1}(A-r_2I)^{m_2}\cdots(A-r_kI)^{m_k} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

along with the fact that generalized eigenvectors share only the zero vector.

So: there is a basis of $\mathbb{R}^n$ consisting of generalized eigenvectors of A (since $m_1+\cdots+m_k=n$)

Since $(\lambda It)((A-\lambda I)t) = ((A-\lambda I)t)(\lambda It)$,

$$\exp(At) = \exp(\lambda It)\exp((A-\lambda I)t)$$
$$= e^{\lambda t}\exp((A-\lambda I)t)$$

If $\vec{u}$ a gen. eigenvector, then

$$\exp(At)\vec{u} = e^{\lambda t}\left(I + (A-\lambda I)t + \frac{1}{2!}(A-\lambda I)^2t^2 + \cdots\right)\vec{u}$$
$$= e^{\lambda t}\left(\vec{u} + t(A-\lambda I)\vec{u} + \frac{1}{2!}t^2(A-\lambda I)^2\vec{u} + \cdots\right)$$

Since $(A-\lambda I)^m\vec{u} = \vec{0}$, can stop with finitely many terms.

For a basis $\vec{u}_1, \cdots, \vec{u}_n$ of $\mathbb{R}^n$ of gen. e.vecs of A,

$e^{At}\vec{u}_1, \cdots, e^{At}\vec{u}_n$ is basis of solutions of $\vec{x}' = A\vec{x}$

and we can calculate $e^{At}\vec{u}_i$ by a method. (Not ad hoc)

<u>ex</u> $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\vec{x}$

$\lambda = 1, 1$

$$\mathrm{Nul}\left(\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^2\right)$$

$$= \mathrm{Nul}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2\right) = \mathrm{Nul}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

basis $\{\vec{e}_1, \vec{e}_2\}$

$$e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^t\left(I\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t(A - I)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

$$e^{At}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^t\left(I\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t(A - I)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= e^t\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} te^t \\ e^t \end{pmatrix}$$

so $\left\{\begin{pmatrix} e^t \\ 0 \end{pmatrix}, \begin{pmatrix} te^t \\ e^t \end{pmatrix}\right\}$ fund. sol. set.