From yesterday:

if A $n\times n$ has n lin. indep eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ with e.vals $\lambda_1, \dots, \lambda_n$ then

$\vec{x}'(t) = A\vec{x}(t)$ has the fundamental solution set $\{\vec{v}_1 e^{\lambda_1 t}, \dots, \vec{v}_n e^{\lambda_n t}\}$.

It is OK if eigenvalues are repeated, so long as there are enough eigenvectors!

<u>ex</u> $\vec{x}' = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}\vec{x}$. $\lambda = 2, 2$

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

<u>not</u> $+ c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} t e^{2t}$

The $te^{\lambda t}$ only comes from non-diagonalizable matrices. (later)

What about complex eigenvalues? Everything still works, but what if we want to ensure real solutions?

<u>Fact</u>: if A has real entries and if λ an eigenvalue with eigenvector $\vec{v}$ (over $\mathbb{C}$, the set of complex numbers) then $\bar{\lambda}$ is an eigenvalue with e.vec. $\bar{\vec{v}}$

$$A\vec{v} = \lambda\vec{v} \Rightarrow \overline{A\vec{v}} = \overline{\lambda\vec{v}} \Rightarrow \bar{A}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

$$\Rightarrow A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}} \quad (\text{since } \bar{A} = A)$$

Recall: $\overline{a+bi} = a - bi$. $\bar{\vec{v}}$ means conjugate of each entry.

So, if λ is not real, there is a second eigenvalue $\bar{\lambda}$ with a second eigenvector $\bar{\vec{v}}$

In this case, $\vec{v}e^{\lambda t}, \bar{\vec{v}}e^{\bar{\lambda}t}$ are in fund. sol. set

Write $\underbrace{\lambda = \alpha}_{\text{"real part"}} + \underbrace{\beta i}_{\text{"imaginary part"}}$ and $\vec{v} = \vec{a} + \vec{b}i$

$\vec{a}, \vec{b} \in \mathbb{R}^n$ (real)

so $\bar{\lambda} = \alpha - \beta i$ and $\bar{\vec{v}} = \vec{a} - \vec{b}i$

Using Euler's identity:

$$\vec{v}e^{\lambda t} = (\vec{a} + \vec{b}i)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
$$= e^{\alpha t}(\vec{a}\cos\beta t - \vec{b}\sin\beta t)$$
$$+ ie^{\alpha t}(\vec{a}\sin\beta t + \vec{b}\cos\beta t)$$

$$\bar{\vec{v}}e^{\bar{\lambda}t} = (\vec{a} - \vec{b}i)e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$
$$= e^{\alpha t}(\vec{a}\cos\beta t - \vec{b}\sin(\beta t))$$
$$- ie^{\alpha t}(\vec{a}\sin\beta t + \vec{b}\cos\beta t)$$

so, $\frac{1}{2}\vec{v}e^{\lambda t} + \frac{1}{2}\bar{\vec{v}}e^{\bar{\lambda}t} = (\vec{a}\cos\beta t - \vec{b}\sin\beta t)e^{\alpha t}$

$\frac{1}{2i}\vec{v}e^{\lambda t} + \frac{1}{2i}\bar{\vec{v}}e^{\bar{\lambda}t} = (\vec{a}\sin\beta t + \vec{b}\cos\beta t)e^{\alpha t}$

these may as well be part of fund. sol. set instead.

Benefit: definitely real

However: more complicated!

Continuing yesterday's example:

$$\vec{x}\,' = \begin{bmatrix} 2a & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

$$\lambda = a \pm i\sqrt{1-a^2} \quad \text{when} \quad a^2 < 1$$

Let's just try finding an eigenvector directly:

$\lambda = a + i\sqrt{1-a^2}$

$$\mathrm{Nul}\begin{bmatrix} 2a-(a+i\sqrt{1-a^2}) & -1 \\ 1 & -(a+i\sqrt{1-a^2}) \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} a-i\sqrt{1-a^2} & -1 \\ 1 & -a-i\sqrt{1-a^2} \end{bmatrix}$$

basis: $\begin{bmatrix} 1 \\ a-i\sqrt{1-a^2} \end{bmatrix}$

so since eigenvector is $\begin{bmatrix} 1 \\ a \end{bmatrix} + i\begin{bmatrix} 0 \\ -\sqrt{1-a^2} \end{bmatrix}$

(real) solution is

$$\vec{x} = c_1 e^{at}\left(\begin{bmatrix} 1 \\ a \end{bmatrix}\cos\left(\sqrt{1-a^2}\,t\right) + \begin{bmatrix} 0 \\ -\sqrt{1-a^2} \end{bmatrix}\sin\left(\sqrt{1-a^2}\,t\right)\right)$$

$$+ c_2 e^{at}\left(\begin{bmatrix} 1 \\ a \end{bmatrix}\sin\left(\sqrt{1-a^2}\,t\right) + \begin{bmatrix} 0 \\ -\sqrt{1-a^2} \end{bmatrix}\cos\left(\sqrt{1-a^2}\,t\right)\right)$$

No matter the a, the "period" is $\frac{2\pi}{\sqrt{1-a^2}}$, however the radius changes over time by e^{at}.

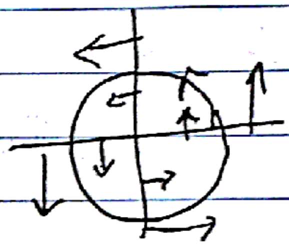
Scanned by CamScanner

Case I: $a = 0$

then solution is $\vec{x} = c_1\left(\begin{bmatrix}1\\0\end{bmatrix}\cos t + \begin{bmatrix}0\\1\end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix}1\\0\end{bmatrix}\sin t - \begin{bmatrix}0\\1\end{bmatrix}\cos t\right)$

$$= \begin{bmatrix}\cos t & \sin t\\ \sin t & -\cos t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

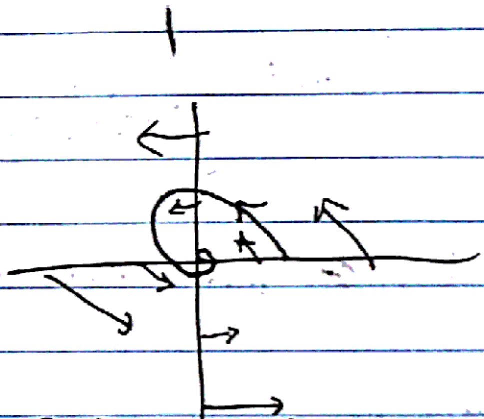
circles.



Case II: $a < 0$.

ex $a = \frac{-1}{2}$

$$\vec{x}' = \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\vec{x}$$

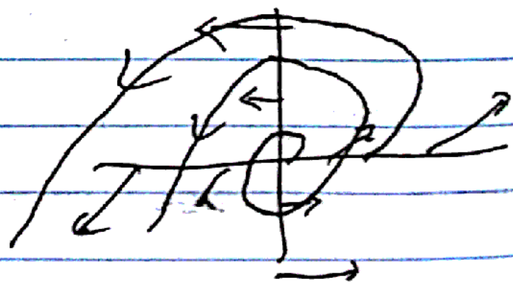


$$\vec{x} = c_1 e^{-\frac{1}{2}t}\left(\begin{bmatrix}1\\1/2\end{bmatrix}\cos\frac{\sqrt{3}}{2}t + \begin{bmatrix}1\\ \sqrt{3}/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)$$
$$+ c_2 e^{-\frac{1}{2}t}\left(\begin{bmatrix}1\\1/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t - \begin{bmatrix}1\\ \sqrt{3}/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)$$

Case III: $a > 0$

ex $a = \frac{1}{2}$

$$\vec{x}' = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\vec{x}$$



$$\vec{x} = c_1 e^{\frac{1}{2}t}\left(\begin{bmatrix}1\\-1/2\end{bmatrix}\cos\frac{\sqrt{3}}{2}t + \begin{bmatrix}1\\ \sqrt{3}/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)$$
$$+ c_2 e^{\frac{1}{2}t}\left(\begin{bmatrix}1\\1/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t - \begin{bmatrix}1\\ \sqrt{3}/2\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)$$

So: real part of e.val controls developing radius.

~~**~~ The sign of the imaginary part controls nothing! They come in conjugate pairs, so this is meaningless.
The direction of rotation is caused by eigenvectors themselves.

Nonhomogeneous systems

$\vec{x}' = A\vec{x} + \vec{f}$

$\uparrow$ nonhomogeneous part

ex $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

guess for particular:

$\vec{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$

$\vec{x}_p' = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$

solve $\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x}_p + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Variation of parameters

solve by ~~guessing~~ solving $\vec{x}_p = X(t)\vec{v}(t)$ for $\vec{v}$

$\vec{x}_p' = X(t)\vec{v}'(t) + X'(t)\vec{v}(t)$

$= AX(t)\vec{v}(t) + \vec{f}$

since $AX(t) = X'(t)$, $\quad X\vec{v}' + AX\vec{v} = AX\vec{v} + \vec{f}$

so $X\vec{v}' = \vec{f}$.

Thus, $\vec{v}' = X^{-1}(t)\vec{f}$

so $\vec{v} = \int X^{-1}(t)\vec{f}(t)\,dt$

ex $\vec{x}'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Matrix exponential

For $\vec{x}' = A\vec{x}$,

$\vec{x} = e^{At}\vec{c}$ is solution

since $\vec{x}' = Ae^{At}\vec{c} = A\vec{x}$.

e^{At} is matrix exponential

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$$

$$\frac{d}{dt}e^{At} = \cdots = Ae^{At}$$