Aug 3

## Homogeneous linear systems with constant coeff.

Consider $A$ an $n \times n$ matrix. Then

$\vec{x}'(t) = A\vec{x}(t)$ is a homog. lin. sys. w/. const. coeff.

Suppose $\vec{v}$ is an eigenvector of $A$ with $A\vec{v} = \lambda\vec{v}$.

Guess: $\vec{v}e^{\lambda t}$ is a solution.

1) $(\vec{v}e^{\lambda t})' = \vec{v}\lambda e^{\lambda t}$

2) $A(\vec{v}e^{\lambda t}) = (A\vec{v})e^{\lambda t} = \lambda\vec{v}e^{\lambda t}$

Indeed, it is a solution.

If $A$ has $n$ (~~distinct~~) eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\vec{v}_1, \ldots, \vec{v}_n$ the eigenvectors,

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$$

is a general solution to $\vec{x}'(t) = A\vec{x}(t)$.

Why? 1) there are $n$ solutions in $\{\vec{v}_1 e^{\lambda_1 t}, \ldots, \vec{v}_n e^{\lambda_n t}\}$

2) they are independent because $\vec{v}_1, \ldots, \vec{v}_n$ are lin. indep (eval. at $t=0$)

By existence & uniqueness, this is a fund. sol. set.

Another way to understand this: $A = PDP^{-1}$ when there is a basis of eigenvectors.

$$\vec{x}' = PDP^{-1}\vec{x} \Rightarrow (P^{-1}\vec{x})' = D(P^{-1}\vec{x})$$

let $\vec{y} = P^{-1}\vec{x}$. $\vec{y}' = D\vec{y}$

Scanned by CamScanner

ex $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$

Char poly: $\det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda I\right)$

$$= \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

e.vals $\lambda = 3, -1$. Distinct so def. diagonalizable

$\lambda = 3$: $\mathrm{Nul}\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3I\right) = \mathrm{Nul}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\lambda = -1$: $\mathrm{Nul}\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + I\right) = \mathrm{Nul}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

basis: $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

so $\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

but: for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & \\ & -1 \end{pmatrix}$ said

by letting $\vec{y} = P^{-1}\vec{x}$ get

$$\vec{y}' = D\vec{y} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 3 & \\ & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

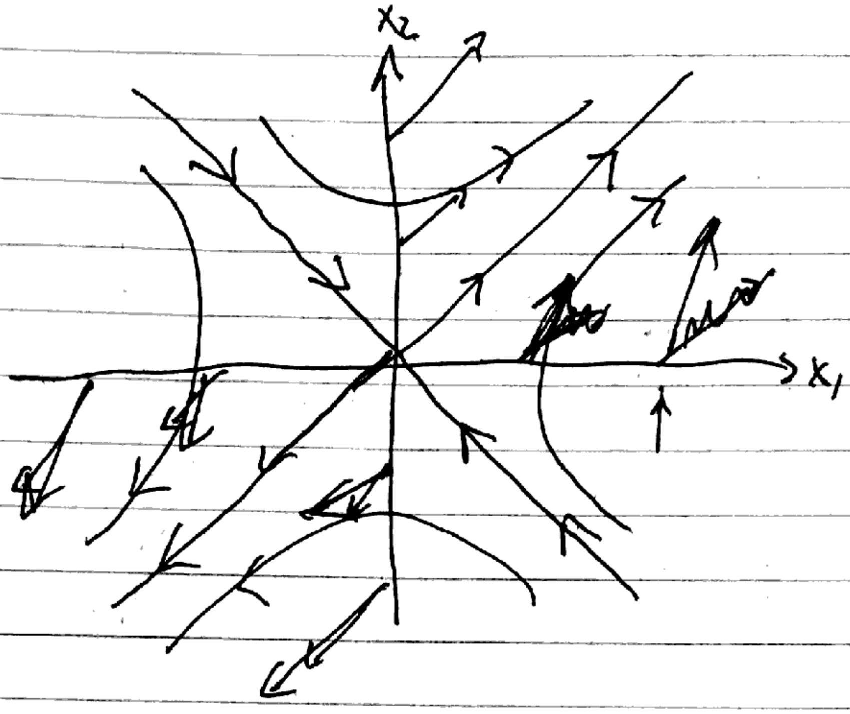
$$\begin{cases} y_1' = 3y_1 \\ y_2' = -y_2 \end{cases}$$

solution: $y_1 = c_1 e^{3t}$, $y_2 = c_2 e^{-t}$

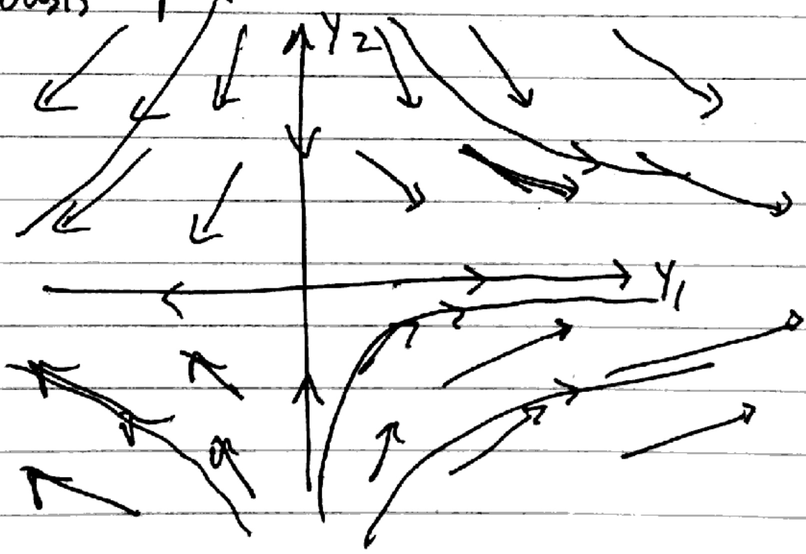
$$\vec{x} = P\begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Scanned by CamScanner




in basis P:




ex $\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}(t)$

char poly $= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$

$= \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$

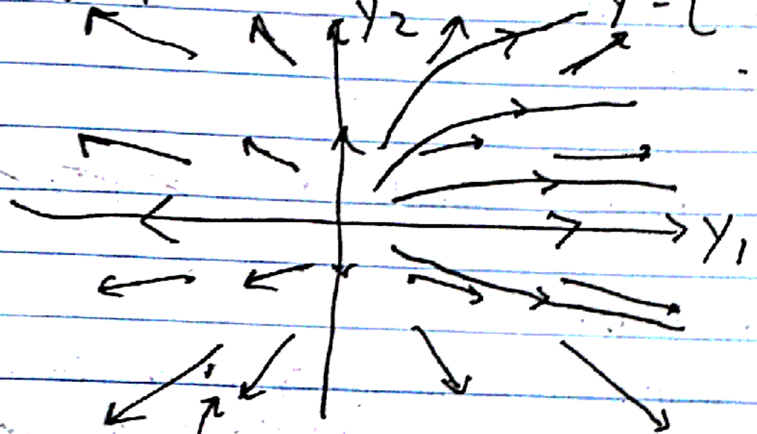
$\lambda = 3$: $\mathrm{Nul}\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

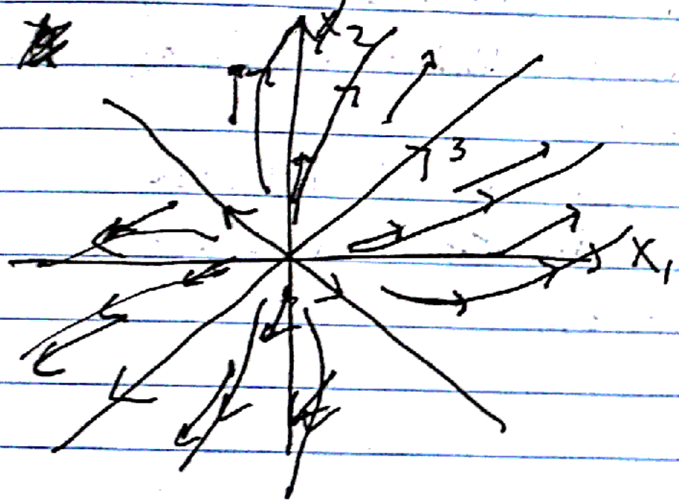
$\lambda = 1$: $\mathrm{Nul}\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Scanned by CamScanner

so $\vec{x}(t) = c_1 \begin{bmatrix}1\\1\end{bmatrix} e^{3t} + c_2 \begin{bmatrix}-1\\1\end{bmatrix} e^{t}$

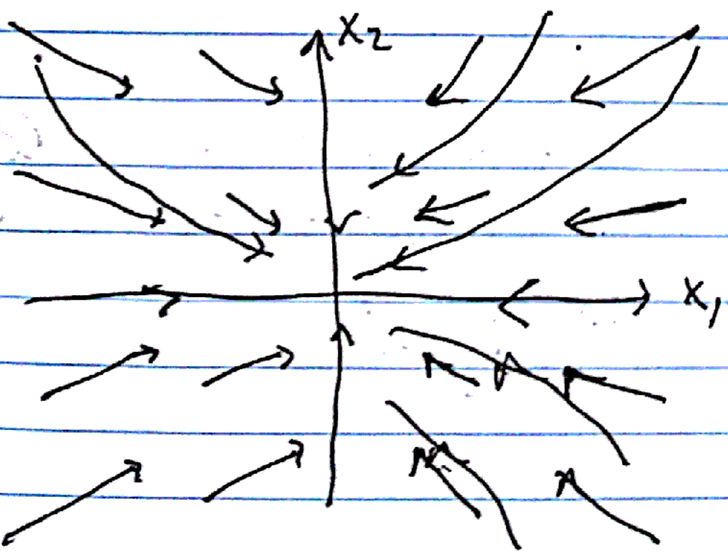
in basis p : $\vec{y}' = \begin{pmatrix}3 & \\ & 1\end{pmatrix}\vec{y}$




ex $\vec{x}'(t) = \begin{bmatrix}-2 & \\ & -1\end{bmatrix}\vec{x}$

$\vec{x} = c_1 \begin{bmatrix}1\\0\end{bmatrix} e^{-2t} + c_2 \begin{bmatrix}0\\1\end{bmatrix} e^{-t}$




Scanned by CamScanner

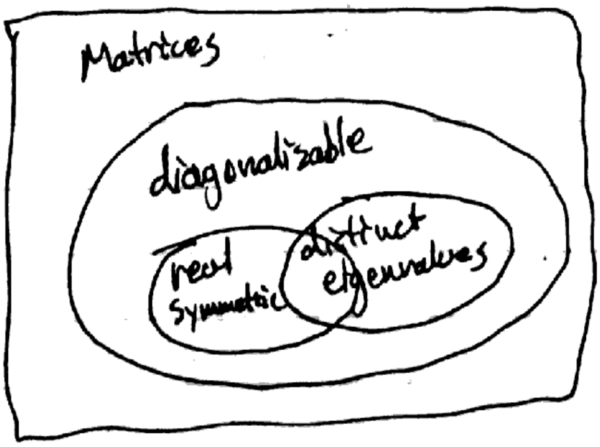
$\begin{pmatrix} a & -1 \\ 1 & \end{pmatrix}$ $\quad (-\lambda)(a-\lambda)+1 = \lambda^2 - a\lambda + 1$

$a^2-4$

Now, we have seen
- 2 positive e.vals
- 1 pos 1 neg
- 2 neg

All that remains would be ~~complex (A~~ non-real

First, some facts:
- If a matrix has $n$ linearly indep. eigenvectors then it is diagonalizable (and the above applies)
- If a matrix has $n$ distinct eigenvalues, it is diagonalizable
- If a matrix is symmetric ($A = A^T$) with real entries, it ~~has~~ is diagonalizable (w/ real eigenvalues)




What about complex eigenvalues? We will use the following toy model

$$\vec{x}'(t) = \begin{bmatrix} 2a & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}$$

Scanned by CamScanner

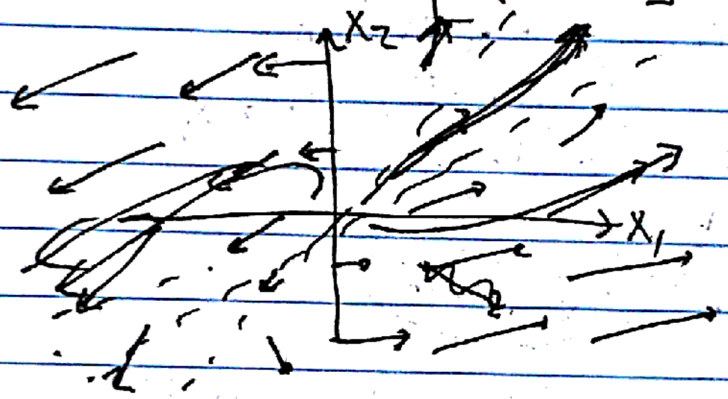
char poly : $\begin{vmatrix} 2a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(2a-\lambda)+1$

$= \lambda^2 - 2a\lambda + 1$

$\lambda = \dfrac{2a \pm \sqrt{(2a)^2-4}}{2} = a \pm \sqrt{a^2-1}$

If $a^2-1 > 0$ then two distinct real roots

<u>ex</u> $a=2$ $\quad \vec{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$



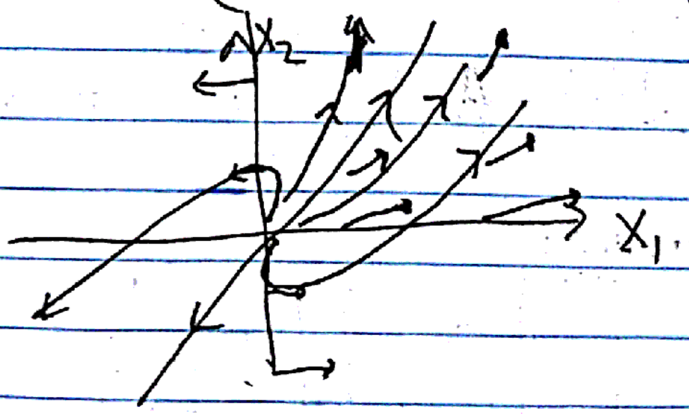

If $a^2-1=0$, then double root.

<u>ex</u> $a=1$ : $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \quad \lambda = 1$

$Nul\begin{pmatrix} 2-1 & -1 \\ 1 & -1 \end{pmatrix} = Nul\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} =$

Basis : $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ (not diagonalizable)

Yet:

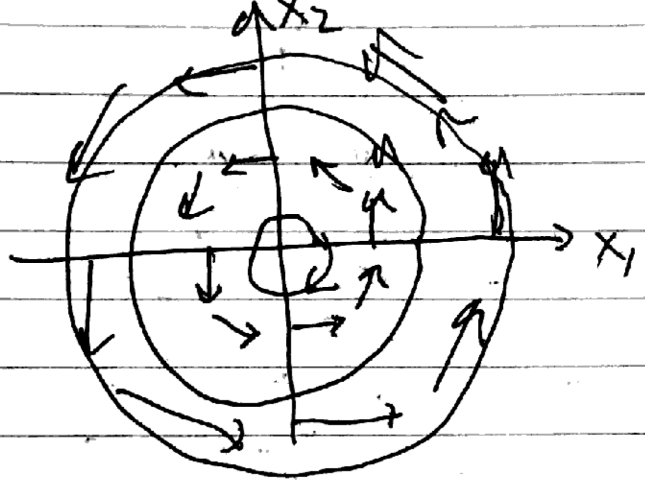


If $a^2-1<0$

roots: $a \pm i\sqrt{1-a^2}$

Case I: $a=0$

$$\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}(t) \cdots$$




e.vals: $\pm i$ (diagonalizable)

$\lambda = i$: $\mathrm{Nul}\begin{pmatrix} 0-i & -1 \\ 1 & 0-i \end{pmatrix}$ basis: $\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$

$\lambda = -i$: $\mathrm{Nul}\begin{pmatrix} 0+i & -1 \\ 1 & 0+i \end{pmatrix}$ basis: $\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$

$$\vec{x}(t) = d_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} + d_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-it}$$

however:

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$$\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

Scanned by CamScanner

so eigenvectors of form $\begin{bmatrix}0\\1\end{bmatrix} \pm \begin{bmatrix}1\\0\end{bmatrix} i$

Book gives real representation

$$\vec{x}(t) = c_1\left(\begin{bmatrix}0\\1\end{bmatrix} e^{0t}\cos t - \begin{bmatrix}1\\0\end{bmatrix} e^{0t}\sin t\right)$$

$$+ c_2\left(\begin{bmatrix}0\\1\end{bmatrix} e^{0t}\sin t + \begin{bmatrix}1\\0\end{bmatrix} e^{0t}\cos t\right)$$

$$= \cancel{c_1 \begin{bmatrix}0\\1\end{bmatrix}\cos t}$$

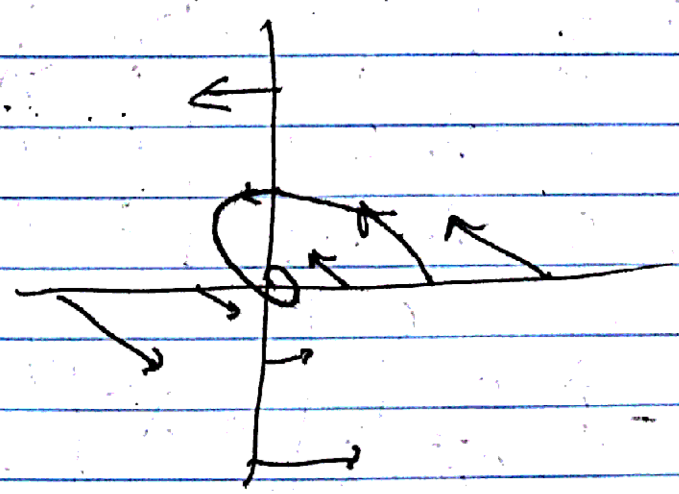
$$= c_1\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} + c_2\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$$

$$= \begin{bmatrix}-\sin t & \cos t\\ \cos t & \sin t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

(periodic, at least!)

Case II : $a < 0$. ex: $a = -1/2$

$$\vec{x}'(t) = \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}\vec{x}(t)$$



spiral inward

Scanned by CamScanner

$(-1-\lambda)(-\lambda)+1 = \lambda^2+\lambda+1$

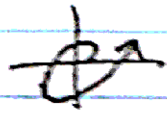
e.vals: $\lambda = -\frac{1}{2} \pm i\sqrt{1-\left(\frac{1}{2}\right)^2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

two evecs of form $\vec{a} \pm i\vec{b}$

$$\vec{x}(t) = c_1\left(\vec{a} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \vec{b} e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$
$$+ c_2\left(\vec{a} e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) + \vec{b} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Case III: $a>0$. ex: $\frac{1}{2}$



ex $\boxed{m}$ —k— $\boxed{m}$

$$\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

let $c = \frac{k}{m}$

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\frac{k}{m} & \frac{k}{m} & -\lambda & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ c & -\lambda & 0 \\ -c & 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -\lambda & 1 \\ -c & c & 0 \\ c & -c & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -c & c & -\lambda & 0 \\ 0 & 0 & -\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & c & -\lambda-\frac{c}{\lambda} & 0 \\ 0 & 0 & -\lambda & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda-\frac{c}{\lambda} & \frac{c}{\lambda} \\ 0 & 0 & -\lambda & -\lambda \end{vmatrix} = \lambda^2 \begin{vmatrix} -\lambda-\frac{c}{\lambda} & \frac{c}{\lambda} \\ -\lambda & -\lambda \end{vmatrix}$$

Scanned by CamScanner