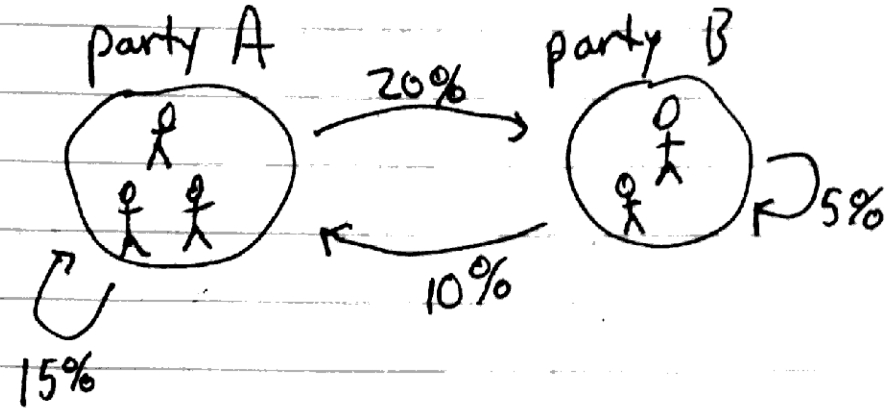
Aug 2

Systems of differential equations

Consider the following situation



Each party, some percentage think they're "too cool" for it and switch to the other, and some percentage have so much fun they manage to invite a friend to come

$$A' = 0.15A + 0.1B - 0.2A = -0.05A + 0.1B$$
$$B' = 0.2A + 0.05B - 0.1B = 0.2A - 0.05B$$

This is a system of two differential equations. If we let $\vec{x}(t) = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$, then we can write it in matrix form

$$\vec{x}\,' = \begin{bmatrix} -0.05 & 0.1 \\ 0.2 & -0.05 \end{bmatrix} \vec{x} \qquad \left(\vec{x}\,'(t) = \begin{bmatrix} A'(t) \\ B'(t) \end{bmatrix} \right)$$

We will use diagonalization and matrix exponentials to solve these kinds of differential equations.

The mass spring system was $mx'' + kx = 0$. We can always create a matrix form for a lin. diff. eq. by introducing new variables. Let $v = x'$, so $mv' + kx = 0$. This is a system:

$$\begin{cases} x' = v \\ v' = -\frac{k}{m}x \end{cases}$$

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

<u>ex</u> $y''' + 2y'' - 3y' + 4y = 0$

Let $y_0 = y$

$y_1 = y'$ so $y_1 = y_0'$

$y_2 = y''$ $\quad y_2 = y_1'$

$$y_2' + 2y_2 - 3y_1 + 4y_0 = 0$$

so $y_2' = -2y_2 + 3y_1 - 4y_0$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 3 & -2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

In general, $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$ can be written as

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$y_i = y^{(i)}$

(with derivs of y treated as separate vars ex  (coupled masses)

let x_1, v_1 be pos/veloc of first mass
x_2, v_2 of second

$$v_1 = x_1'$$
$$v_2 = x_2'$$
$$m v_1' = k(x_2 - x_1)$$
$$m v_2' = k(x_1 - x_2)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

So far, these are all <u>homogeneous</u>. A general linear diff. eq. is written as

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t) \quad \text{(\underline{normal} form)}$$

(Yes, the matrix A may be a function of time!)

<u>ex</u> $x'' + x = \cos(\omega t)$ (driven mass-spring system)

$v = x'$ so $v' = -x + \cos(\omega t)$

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(\omega t) \end{bmatrix}$$

<u>Thm</u> (Existence and uniqueness) Suppose $A(t)$ and $f(t)$ continuous on an open interval I containing t_0. Given an initial condition $\vec{x}_0$, there is a unique solution $\vec{x}(t)$ with domain I satisfying $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$ and $\vec{x}(t_0) = \vec{x}_0$.

Let us define the linear operator

$$T(\vec{x}(t)) = \vec{x}'(t) - A(t)\vec{x}(t)$$

$$\ker T = \{ \vec{x}(t) \mid \vec{x}'(t) - A(t)\vec{x}(t) = 0 \}$$

That is, the kernel is the set of solutions to the homogeneous equa

Existence & uniqueness says for any $\vec{f}(t)$, $T(\vec{x}(t)) = \vec{f}$ has a solution. It is not unique unless we also add the constraint $\vec{x}(t_0) = \vec{x}_0$.

Suppose $\vec{x}_p$ ~~is a solution~~ satisfies $T(\vec{x}_p) = f$ and $\vec{x}_h$ in $\ker T$ (ie., satisfies $T(\vec{x}_h) = 0$).

Then $T(\vec{x}_p + \vec{x}_h) = f + 0 = f$

so $\vec{x}_p + \vec{x}_h$ also a solution.

~~so $(\vec{x}_p + \vec{x}_h)' - A(t)(\vec{x}_p + \vec{x}_h) = f$~~

~~or $\vec{x}_p + \vec{x}_h$~~

~~Since $\ker T$ is a subspace,~~

The map $\ker T \longrightarrow \mathbb{R}^n$ is a

$$\vec{x}_h \longmapsto \vec{x}_h(t_0)$$

linear transformation. Exist & unique says this is an isomorphism, so $\ker T$ is n-dimensional. The n-element basis to this is called the fundamental solution set.

"Vector functions $\vec{x}_1, \cdots, \vec{x}_n$ are <u>linearly independent</u> on I if

$c_1, \cdots, c_n$ are such that $c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \vec{0}$ for all $t \in I$, then $c_1 = \cdots = c_n = 0$

(or: $(\vec{x}_1(t) \cdots \vec{x}_n(t))\vec{c} = \vec{0}$ has only the trivial solution)

weird book notation: $\operatorname{col}(e^t, 0, e^{-t}) = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}$

<u>def</u> The <u>Wronskian</u> of $\vec{x}_1, \cdots, \vec{x}_n$ is

$$W[\vec{x}_1, \cdots, \vec{x}_n](t) = \det\begin{pmatrix} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{pmatrix}$$

if $\vec{x}_1, \cdots \vec{x}_n$ dependent, then $W[\vec{x}_1, \cdots, \vec{x}_n](t) = 0$ for all t.

but: for $\vec{x}_1 = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} t|t| \\ 2|t| \end{bmatrix}$

independent: if $c_1\vec{x}_1 + c_2\vec{x}_2 = 0$

@ $t=1$, $c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \Rightarrow c_1 = -c_2$

@ $t=-1$ $c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0 \Rightarrow c_1 = c_2$

so $c_1 = c_2 = 0$

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} t^2 & t|t| \\ 2t & 2|t| \end{vmatrix} = 2|t|t^2 - 2t^2|t| = 0$$

so converse not true.

but: if $\vec{x}_1, \dots, \vec{x}_n$ solutions to $\vec{x}' = A\vec{x}$

and $W[\vec{x}_1, \dots, \vec{x}_n](t_0) = 0$ for ~~all~~ some t_0

then there are $c_1, \dots, c_n$ with

$$c_1 \vec{x}_1(t_0) + \dots + c_n \vec{x}_n(t_0) = \vec{0}$$

$\vec{0}$ is a solution to $\vec{x}' = A\vec{x}$ with init. cond. $\vec{0}$

so by uniqueness, (since $c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$ a soln)

$$c_1 \vec{x}_1(t) + \dots + c_n \vec{x}_n(t) = \vec{0} \quad \text{for all } t$$

so $\vec{x}_1, \dots, \vec{x}_n$ dependent.

summarized: if $\vec{x}_1, \dots, \vec{x}_n$ solutions to $\vec{x}' = A\vec{x}$, either

- dependent and Wronskian always 0
- independent and Wronskian never 0

We used the fact that lin. combs. of homog. solns is a soln. Given $\{\vec{x}_1, \dots, \vec{x}_n\}$ fund. soln set of $\vec{x}' = A\vec{x}$, the fundamental matrix is

$$X(t) = \left(\vec{x}_1(t) \ \cdots \ \vec{x}_n(t)\right) \quad n \times n$$

every solution is then of the form

$$\vec{x}(t) = X(t)\vec{c} \quad \text{for some } \vec{c} \in \mathbb{R}^n.$$

If $\vec{x}_p$ is a solution to $\vec{x}' = A\vec{x} + \vec{f}$, then every solution is then of the form

$$\vec{x}(t) = \vec{x}_p(t) + X(t)\vec{c} \quad \text{for some } \vec{c} \in \mathbb{R}^n$$

An ~~general~~ initial condition $\vec{x}_0$ satisfies then

$$\vec{x}_0 = \vec{x}_p(t_0) + X(t_0)\vec{c}$$

$$|X(t_0)| \neq 0, \quad \text{so} \quad \vec{c} = X(t_0)^{-1}(\vec{x}_0 - \vec{x}_p(t_0)).$$

So, given a particular solution and an init. cond, always solvable!

(so, if you have fund. sol. set (& part.) can write gen. soln)

If A is a constant matrix with n distinct e. vals $r_1, \ldots, r_n$, $\vec{x}' = A\vec{x}$ has fund. sol set

$$\{e^{r_1 t}\vec{u}_1, \ldots, e^{r_n t}\vec{u}_n\}$$

where $\vec{u}_1, \ldots, \vec{u}_n$ are eigenvectors.

so gen solution is $\vec{x}(t) = (e^{r_1 t}\vec{u}_1 \cdots e^{r_n t}\vec{u}_n)\vec{c}$

$$= c_1 e^{r_1 t}\vec{u}_1 + \cdots + c_n e^{r_n t}\vec{u}_n$$

or $= P\begin{bmatrix} c_1 e^{r_1 t} \\ \vdots \\ c_n e^{r_n t} \end{bmatrix}$ $\quad (P = (\vec{u}_1 \cdots \vec{u}_n))$

Why? $A = PDP^{-1}$

$$\vec{x}' = PDP^{-1}\vec{x} \qquad (P^{-1}\vec{x})' = P^{-1}\vec{x}'$$

$$(P^{-1}\vec{x})' = D(P^{-1}\vec{x}) \qquad \text{let } \vec{y} = P^{-1}\vec{x}$$

$\vec{y}' = D\vec{y}$. Easy to solve!

then $\vec{x} = P\vec{y}$

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ex $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$

$\lambda = 3, 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ex $\vec{x}' = \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \vec{x}$

this is $x_1' = 3x_1 \Rightarrow x_1 = c_1 e^{3t}$
$x_2' = x_2 \qquad x_2 = c_2 e^{t}$

so $\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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1) Set up in normal form

a) $y'' + 2y' + y = 0$

b) $y'' + 2y' + y = \cos(\omega t)$

c) $\begin{cases} x' = 2x - 3y \\ y' = -3x + 4y \end{cases}$

d) $\begin{cases} x'' + x + y = 0 \\ y'' + y' + x = 0 \end{cases}$

e) $\begin{cases} x' = (\sin t)x + e^{t}y \\ y' = (\cos t)x + (a + t^3)y \end{cases}$