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Homogeneous lin. diff. eqns w/ constant coeff.

A homog. lin. diff. eqn w/ constant coeff is
$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y_0 = 0$$
for $a_0, \ldots, a_{n-1} \in \mathbb{R}$. Existence and Uniqueness thm says there are n lin. indep. solutions with domain $(-\infty, \infty)$

By experience, e^{rt} is probably a solution.
$$\frac{d^k}{dt^k} e^{rt} = r^k e^{rt}$$

so
$$\begin{aligned}(e^{rt})^{(n)} &+ a_{n-1}(e^{rt})^{(n-1)} + \cdots + a_1(e^{rt})' + a_0 e^{rt} \\ &= r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} \\ &= (r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rt}\end{aligned}$$
Since e^{rt} is never zero, then to be a solution, $r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$.
That is, r must be a root of the auxiliary poly
$$P(r) = r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$$
which has n roots with multiplicity.

Hints for finding roots:
1) integer roots are $\pm$divisors of a_0
2) roots of P' come between roots of P
3) odd-degree polys have ≥ 1 real root. (If it isn't rational, it's probably hopeless to actually find roots analytically!)
4) a multiple root of P is a root of P', too.

ex General solution to $y''' - 2y'' - 5y' + 6y = 0$.

$r^3 - 2r^2 - 5r + 6 = 0$

1, 2, 3, 6 divisors of 6.

1 appears to be root.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array} \qquad \begin{aligned} &(r-1)(r^2 - r - 6) \\ &= (r-1)(r-3)(r+2) \\ &r = 1, 3, -2 \end{aligned}$$

$$y = C_1 e^{t} + C_2 e^{3t} + C_3 e^{-2t}$$

found 3 linearly indep. ~~function~~ solutions, so done

ex $y''' + y' - 10y = 0$ $\qquad r^3 + r - 10 = 0$

$$(r-2)(r^2+2r+5) \qquad \begin{array}{r} r^3 + 2r^2 + 5r \\ -2r^2 - 4r - 10 \\ \hline r^3 + r - 10 \end{array}$$

1, 2, 5, 10 divisors, and 2 a root

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$= (r-2)(r^2 + 2r + 5)$

$$\frac{-2 \pm \sqrt{4 - 4\cdot 5}}{2} = -1 \pm 2i$$

$2, -1 \pm 2i$ roots

$$y = C_1 e^{2t} + C_2 e^{-t} \cos 2t + C_3 e^{-t} \sin 2t$$

Are $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ really linearly indep.?
(with $\lambda_1, \ldots, \lambda_n$ distinct)

Method I: Wronskian

$$W[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}]$$

$$= \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \cdots \\ \lambda_1^2 e^{\lambda_1 t} & \lambda_2^2 e^{\lambda_2 t} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

~~[illegible]~~ @ $t_0 = 0$,

$$W[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}](0) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

Vandermonde determinant

$$= (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \cdots (\lambda_n - \lambda_1) \cdot (\lambda_3 - \lambda_2) \cdots (\lambda_n - \lambda_2) \cdots (\lambda_n - \lambda_{n-1})$$

$\neq 0$ if $\lambda_1, \ldots, \lambda_n$ distinct

so they are independent.

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Method II: annihilator. Suppose $c_1 e^{\lambda_1} + \cdots + c_n e^{\lambda_n} = 0$

consider $A = (\frac{d}{dt} - \lambda_2)(\frac{d}{dt} - \lambda_3) \cdots (\frac{d}{dt} - \lambda_n)$

$$A e^{\lambda t} = (\lambda - \lambda_2)(\lambda - \lambda_3) \cdots (\lambda - \lambda_n) e^{\lambda t}$$

Why? The aux. poly for $Ay = 0$ is

$(r - \lambda_2) \cdots (r - \lambda_n)$

ex $(\frac{d}{dt} - 2)(\frac{d}{dt} - 3) y$

$= (\frac{d}{dt} - 2)(y' - 3y)$

$= (y'' - 3y') - 2(y' - 3y)$

$= y'' - 5y' + 6y$

$r^2 - 5r + 6 = (r-2)(r-3)$

So by before,

$(e^{\lambda t})'' - 5(e^{\lambda t})' + 6e^{\lambda t} = (\lambda - 2)(\lambda - 3) e^{\lambda t}$

So, for $\lambda = \lambda_2, \cdots, \lambda_n$, $A e^{\lambda t} = 0$

for $\lambda = \lambda_1$, $A e^{\lambda t} = \underbrace{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_n)}_{\neq 0} e^{\lambda t}$

hence $A(c_1 e^{\lambda_1} + \cdots + c_n e^{\lambda_n}) = A0$

gives $c_1 A(e^{\lambda_1}) = 0$

so $c_1 = 0$.

Can annihilate over and over to get $c_2 = 0, \cdots$

so $c_1 = \cdots = c_n = 0$. Independent!

What about repeated roots?

If r_i has mult m, then

$$e^{r_i t}, \; t e^{r_i t}, \; \dots, \; t^{m-1} e^{r_i t} \text{ indep.}$$

(proof omitted)

ex $y'''' - y''' - 3y'' + 5y' - 2y = 0$

$r^4 - r^3 - 3r^2 + 5r - 2$

maybe $\pm 1, \pm 2$?

1 root.

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & -3 & 5 & -2 \\ & & 1 & 0 & -3 & 2 \\ \hline & 1 & 0 & -3 & 2 & 0 \end{array}$$

$r^3 - 3r + 2$ — 1 root again

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$r^2 + r - 2 = (r+2)(r-1)$

$(r-1)^3(r+2)$

$r = 1, 1, 1, -2$

$$y = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + C_4 e^{-2t}$$

ex part soln to $y'''' - y''' - 3y'' + 5y' - 2y = e^t$?

$y_p = C t^3 e^t$ will work

$(r^2+2r+5)^2$

$$\begin{array}{r} 5r^2 + 10r + 25 \\ 2r^3 + 4r^2 + 10r \\ r^4 + 2r^3 + 5r^2 \\ \hline r^4 + 4r^3 + 14r^2 + 20r + 25 \end{array}$$

<u>ex</u> $y'''' + 4y''' + 14y'' + 20y' + 25y = 0$

$$r^4 + 4r^3 + 14r^2 + 20r + 25 = (r^2+2r+5)^2$$

so $r = -1 \pm 2i,\ -1 \pm 2i$

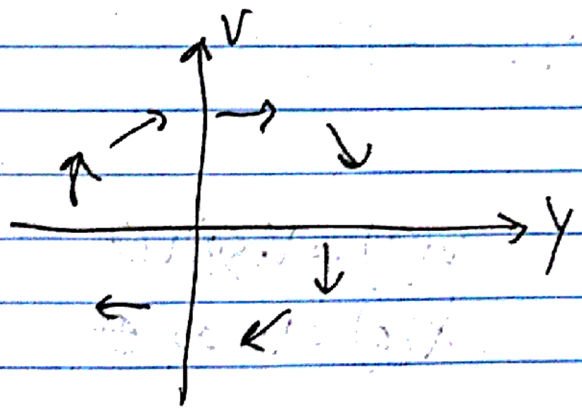
$$y = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + C_3 t e^{-t}\cos 2t + C_4 t e^{-t}\sin 2t$$

<u>Matrix methods</u>

$y' = v$

$mv' = -ky$

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}$$



$dx/dt = -4x + 2y$

$dy/dt = 4x - 4y$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$