July 29

Higher-order lin. diff. eqs

A *linear differential equation of order n* is of the form

$$(*) \quad y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = f(t)$$

where $p_0, \ldots, p_{n-1}, f$ are functions continuous on some interval I. If they are constants, then we say it has *constant coefficients*.

Thm (Existence and uniqueness) Suppose $p_0, \cdots, p_{n-1}, f$ are all continuous on interval (a,b). There is a unique solution to $(*)$ satisfying an initial condition at $t_0 \in (a,b)$

$$\begin{bmatrix} y(t_0) \\ y'(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

with domain (a,b).

A *homogeneous* lin. diff. eqn is when $f=0$. Then, the above is an isomorphism between solutions to the eqn and $\mathbb{R}^n$ (the vectors being interp. as init. conditions at t_0). Why is solution set a subspace?

Let $L = \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t)$.

Then $Ly = 0$ is homog. lin. diff. eqn.

One can check

i) $L(y_1+y_2) = Ly_1 + Ly_2$

ii) $L(cy_1) = c\,Ly_1$

so L is linear. Thus, $\ker L = \{$homog. solns to $L\}$ is a subspace (of the v.space of functions)

Every subspace has a basis, and the isomorphism says it is n-dimensional, so it is just a matter of finding them all! To remind basis def:

def A set of functions $f_1, \dots, f_n$ is lin. indep if whenever $c_1 f_1(t) + \dots + c_n f_n(t) = 0$ for all t, then the constants $c_1, \dots, c_n$ are all 0

Alt: if none is a lin. comb. of the others.

The are dependent otherwise. Alt: if any one is a lin. comb. of the others.

def $f_1, \dots, f_n$ span solution set if every sol. is lin. comb of these. basis is lin. indep. spanning set

If $y_1, \dots, y_n$ solutions to n^{th}-order diff. eq., and if $c_1 y_1 + \dots + c_n y_n = 0$, then

$$c_1 y_1' + \dots + c_n y_n' = 0$$
$$\vdots$$
$$c_1 y_1^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$$

so
$$\begin{bmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \vec{c} = \vec{0}$$

So, if $\vec{c} \neq \vec{0}$, then (i.e., $y_1, \cdots, y_n$ dependent)

$$W[y_1, \cdots, y_n](t) = \begin{vmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix} = 0$$

This is the Wronskian

Suppose conversely the Wronskian is 0 at some t_0. Then the init. cond. vectors are dependent.
So by existence and uniqueness, the corresponding funcs are dependent, too.

So: if $y_1, \cdots, y_n$ solutions to nth-order homog. $Ly = 0$ on (a,b),
TFAE:
1) $y_1, \cdots, y_n$ lin. indep
2) $W[y_1, \cdots, y_n](t_0) \neq 0$ for some t_0
3) $W[y_1, \cdots, y_n](t) \neq 0$ for all t

Contrapositive: TFAE: (remember: nth order solns)
1) $y_1, \cdots, y_n$ lin. dep.
2) $W[y_1, \cdots, y_n](t_0) = 0$ for some t_0
3) $W[y_1, \cdots, y_n](t) = 0$ for all t

basis or fundamental solution set

ex $\{1, x, x^2, x^3, \cdots, x^n\}$ independent.
$y^{(n+1)} = 0$, all solutions to this

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \cdots & n(n-1)x^{n-2} \\ \vdots & & \vdots & & & \vdots \end{vmatrix} \neq 0 \text{ ever}$$

(before, we had to use fund. thm of algebra!)

ex $\{1, \cos x, \sin x, \dots, \cos(nx), \sin(nx)\}$ indep

ex $\{e^{a_1 x}, e^{a_2 x}, \dots\}$ a_i's distinct indep.

$W[\dots](0)$ is a vandermond matrix's det. $\neq 0$.

Homog. lin diff eqns w/ constant coeff.

It is just like before.

1) find auxiliary polynomial's roots
2) write terms corresponding to roots, adjusting for multiplicity.

ex $y''' - 4y'' + 4y' = 0$

$r^3 - 4r^2 + 4r = r(r-2)^2$

$r = 0, 2, 2$

$y = Ae^{0t} + Be^{2t} + Cte^{2t}$

ex $y''' = 0$

$r^3 = 0 \qquad r = 0, 0, 0$

$y = Ae^{0t} + Bte^{0t} + Ct^2e^{0t}$

$= A + Bt + Ct^2$ (quadratics!)

ex $y''' - y'' + y' - y = 0$

$r^3 - r^2 + r - 1 = (r-1)(r^2+1)$

$r = -1, \pm i$

$y = Ae^{-t} + B\cos t + C\sin t$

What is happening is this. For $L = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots a_0$ we can factor it as $L = \left(\frac{d}{dt} - \lambda_1\right)^{n_1}\left(\frac{d}{dt} - \lambda_2\right)^{n_2}\cdots$

where n_i is the mult. of root λ_i in auxiliary poly. Turns out just need to solve each $\left(\frac{d}{dt} - \lambda_i\right)^{n_i} y = 0$ individually and then just add up all solutions!