July 28

Superposition "to place over" aka. linearity

Recall: If A is a matrix and $A\vec{x_1} = \vec{b_1}$ and $A\vec{x_2} = \vec{b_2}$, then $A(c_1\vec{x_1} + c_2\vec{x_2}) = c_1\vec{b_1} + c_2\vec{b_2}$. $\vec{b_2} = \vec{0}$ is the special case of adding a homogeneous solution to a particular. So: to get a solution to a linear comb. of $\vec{b_1}$ and $\vec{b_2}$, linearly combine $\vec{x_1}$ and $\vec{x_2}$ in the same way. More generally, this is just if $AX = B$, then $A(X\vec{c}) = B\vec{c}$.

Linear diff. eqs, too, have this property.

ex $\quad y'' + 3y' + 2y = e^{4t} + e^{-t}$

i) solve $y'' + 3y' + 2y = e^{4t}$

$r = -1, -2 \qquad r = 4$

guess $y_p = Ae^{4t}$

$y_p' = 4Ae^{4t}$

$y_p'' = 16Ae^{4t}$

$16Ae^{4t} + 12Ae^{4t} + 2Ae^{4t} = e^{4t}$

$30Ae^{4t} = e^{4t} \Rightarrow A = \frac{1}{30}$

ii.5) $y'' + 3y' + 2y = 0$

$y = Ae^{-t} + Be^{-2t}$

ii) solve $y'' + 3y' + 2y = e^{-t}$

$r = -1$

guess $y_p = Ate^{-t}$

$y_p' = -Ate^{-t} + Ae^{-t}$

$y_p'' = Ate^{-t} - 2Ae^{-t}$

$(Ate^{-t} - 2Ae^{-t}) + 3(-Ate^{-t} + Ae^{-t}) + 2(Ate^{-t}) = e^{-t}$

$-2A \qquad +3A \qquad = 1$

$\Rightarrow A = 1$

iii) solve $y'' + 3y' + 2y = e^{4t} + e^{-t}$

$y = \frac{1}{30}e^{4t} + te^{-t} + Ae^{-t} + Be^{-2t}$

Thm (Existence and uniqueness). if $y'' + ay' + by = f(t)$ is a diff. eq. with a solution y_p, and if t_0, y_0, y_1 are constants, then there is a unique solution y with $y(t_0) = y_0$ and $y'(t_0) = t_1$.

Pf there is a unique solution y_q to $y'' + ay' + by = 0$ with $\begin{bmatrix} y_q(t_0) \\ y_q'(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} - \begin{bmatrix} y_p(t_0) \\ y_p'(t_0) \end{bmatrix}$.

By superposition, $y = y_q + y_p$ is a solution to $y'' + ay' + by = f$, and $y_0 = y_q(t_0) + y_p(t_0)$ and $y_1 = y_q'(t_0) + y_p'(t_0)$.

Conversely, the difference between any two solutions is a homogeneous solution with init. conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the difference is the 0 function. ∎

This means that even with a "driving term" solutions for a particular initial condition is unique. If no particular solution exists, the equation has no solutions. (but this basically never happens).

ex $y'' + y = \cos(\omega t)$

$r = \pm i \qquad r = \pm \omega i$

Two cases:

1) $|\omega| \neq 1$. Then $y_p = A \cos \omega t + B \sin \omega t$

$y_p' = -A\omega \sin \omega t + B\omega \cos \omega t$

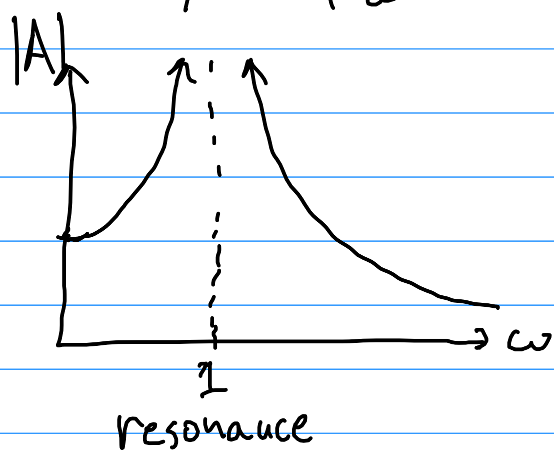
$y_p'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$

so $\begin{cases} -A\omega^2 + A = 1 \\ -B\omega^2 + B = 0 \end{cases}$

$$A = \frac{1}{1-\omega^2} \qquad B = 0$$

$$y_p = \frac{1}{1-\omega^2}\cos(\omega t)$$

general: $y = \frac{1}{1-\omega^2}\cos(\omega t) + C\cos t + D\sin t$



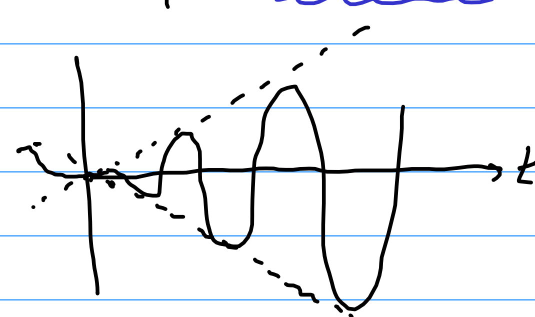
(2) $|\omega| = 1$

$$y_p = At\cos t + Bt\sin t$$
$$y_p' = -At\sin t + A\cos t + Bt\cos t + B\sin t$$
$$y_p'' = -At\cos t - 2A\sin t - Bt\sin t + 2B\cos t$$

$\cos t$: $2B = 1$ $B = \frac{1}{2}$

$\sin t$: $-2A = 0$ $A = 0$

solution: $y = \frac{1}{2}t\cos t + C\cos t + D\sin t$



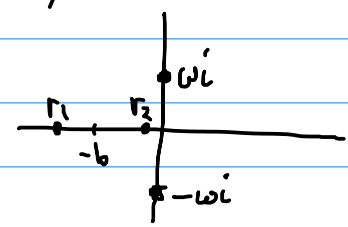
Larger and larger amplitudes at resonance!

ex Let's add dampening: $y'' + 2by' + y = \cos \omega t$

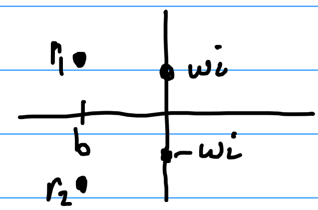
$$r = \frac{-2b \pm \sqrt{4b^2 - 4}}{2} = -b \pm \sqrt{b^2 - 1} \qquad \pm \omega i$$

Now there is no way for $-b \pm \sqrt{b^2-1} = \pm \omega i$

For $b^2 \geq 1$,



For $b^2 < 1$,



$$y_p = A\cos \omega t + B \sin \omega t$$
$$y_p' = -A\omega \sin \omega t + B\omega \cos \omega t$$
$$y_p'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$\cos$: $(-A\omega^2) + 2b(B\omega) + A = 1$

$\sin$: $(-B\omega^2) + 2b(-A\omega) + B = 0$

$$\begin{cases} (1-\omega^2)A + 2b\omega B = 1 \\ -2b\omega A + (1-\omega^2)B = 0 \end{cases}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1-\omega^2 & 2b\omega \\ -2b\omega & 1-\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(1-\omega^2)^2 + 4b^2\omega^2} \begin{bmatrix} 1-\omega^2 \\ 2b\omega \end{bmatrix}$$

$$y_p = \frac{1}{(1-\omega^2)^2 + 4b^2\omega^2}\left((1-\omega^2)\cos \omega t + 2b\omega \sin \omega t\right)$$

Notice: as $b \to 0$, we get prev. solution

Fact: $A\cos \omega t + B \sin \omega t = \underbrace{\sqrt{A^2+B^2}}_{\text{amplitude}} \cos(\omega t - k)$ for some k.

The amplitude of the above is

$$\frac{1}{\sqrt{(1-\omega^2)^2+4b^2\omega^2}}$$

peak is when $\frac{d}{d\omega}\left((1-\omega^2)^2+4b^2\omega^2\right)=0$

$$\Updownarrow$$

$$2(1-\omega^2)(-2\omega) + 8b^2\omega = 0$$

one critical point at $\omega=0$ (ignore)

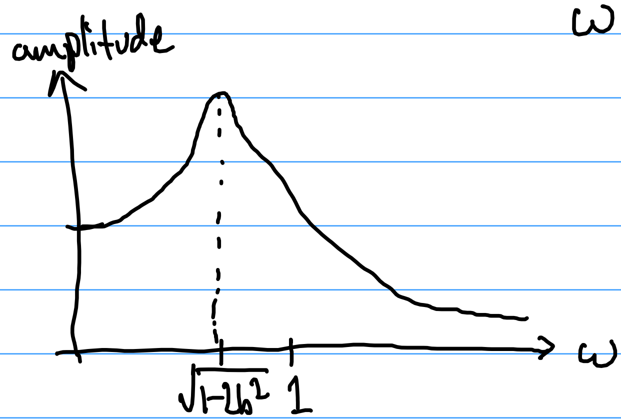
now $2(1-\omega^2)(-2) + 8b^2 = 0$

$$-4 + 4\omega^2 + 8b^2 = 0$$

$$\omega^2 = 1-2b^2$$

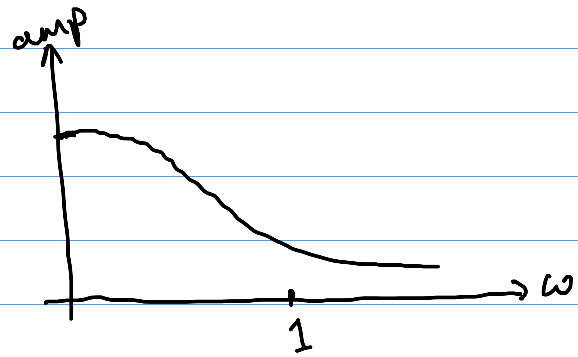
$$\omega = \pm\sqrt{1-2b^2}$$

$(2b^2<1)$



as $b\to 0$, peak $\to 1$ and becomes an asymptote.

$2b^2>1$



no peak at all! too much dampening for driver to do anything at any frequency!

Variation of parameters

If $y'' + ay' + by = 0$ has y_1, y_2 lin. indep solutions, then $y'' + ay' + by = f$ has particular solution

$$y_p = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$$

with $$v_1(t) = \int \frac{-f(t)\, y_2(t)\, dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

and $$v_2(t) = \int \frac{f(t)\, y_1(t)\, dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

(that is, $v_1 = \int \frac{-fy_2\, dt}{W[y_1, y_2]}$ and $v_2 = \int \frac{-fy_1\, dt}{W[y_1, y_2]}$)

The book gives the derivation, but there is no need to do it over and over. Read it, though.

ex $y'' - y = \tan t$

$r = \pm 1$

e^{-t}, e^{t} $\quad v_1 = \int \frac{-\tan(t)\, e^{t}\, dt}{W[e^{-t}, e^{t}]}$ $\quad v_2 = \int \frac{\tan(t)\, e^{-t}\, dt}{W[e^{-t}, e^{t}]}$