Complex roots

July 27

Suppose $y'' + ay' + by = 0$ has an auxiliary eqn with a complex root $\alpha + \beta i$. This happens, recall, when $a^2 - 4b < 0$. Then, the diff.eq. has solutions

$$y = A e^{\alpha t} \cos \beta t + B e^{\alpha t} \sin \beta t .$$

(this is just a transformation of $Ce^{(\alpha+\beta i)t} + De^{(\alpha-\beta i)t}$ which guarantees real-valued results.)

Why? version 1

If you substitute these into the diff.eq., you can (after some calculation!) see they are solutions. Also, you can use the Wronskian to demonstrate independence. The dim. of sol. space is 2, so that's it.

Why? Version 2

Let $z = \alpha + \beta i$. What does e^{zt} mean as a solution? Let's believe $\frac{d}{dt} e^{zt} = z e^{zt}$, even for complex numbers, Let's show $e^z = e^\alpha (\cos\beta + i \sin\beta)$

which justifies it as a solution.

We saw before that $\mathbb{C}$ can be thought of as 2×2 matrices with the correspondence

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \longleftrightarrow \alpha + \beta i$$

add./mult. of matrices corr. to add/mult of complex numbers. ("field isomorphism")

We define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z)$, the Taylor series.

We compute

$$\exp(z) = \exp\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^n$$

Since $\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}\begin{pmatrix} & -\beta \\ \beta & \end{pmatrix} = \begin{pmatrix} \alpha\beta & \\ & -\alpha\beta \end{pmatrix} = \begin{pmatrix} & -\beta \\ \beta & \end{pmatrix}\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} + \begin{pmatrix} & -\beta \\ \beta & \end{pmatrix}\right)^n = \cdots = \left(\sum_{n=0}^{\infty} \frac{1}{n!}\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!}\begin{pmatrix} & -\beta \\ \beta & \end{pmatrix}^n\right)$$

$$= \begin{pmatrix} e^\alpha & \\ & e^\alpha \end{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} & -\beta \\ \beta & \end{pmatrix}^n$$

Now, $\frac{1}{0!}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} & -\beta \\ \beta & \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -\beta^2 & \\ & -\beta^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} & \beta^3 \\ -\beta^3 & \end{pmatrix} + \cdots$

$$= \cdots = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

so $\exp\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = e^\alpha \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

Euler's formula

Which, under the correspondence, says $\boxed{e^{\alpha+\beta i} = e^\alpha(\cos\alpha + i\sin\beta)}$

To check: $e^{\alpha+0\beta} = e^\alpha(\cos 0 + i\sin 0) = e^\alpha$, so it is the normal exponential for reals.

Fact about polynomials with real coefficients: if $z = \alpha+\beta i$ is a root, so is $\bar{z} = \alpha + \beta i$.

So, e^{zt} and $e^{\bar{z}t}$ are both solutions.

$$Ae^{zt} + Be^{\bar{z}t} = Ae^{\alpha t}(\cos\beta t + i\sin\beta t) + Be^{\alpha t}(\cos\beta t - i\sin\beta t)$$

sin is odd

$$= (A+B)e^{\alpha t}\cos\beta t + (Ai - Bi)e^{\alpha t}\sin\beta t$$

Is it true that $\{e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t\}$ is a basis? Suppose we want to make

$$Ce^{\alpha t}\cos\beta t + De^{\alpha t}\sin\beta t.$$

Then, solve $\left[\begin{array}{cc|c} 1 & 1 & C \\ i & -i & D \end{array}\right]$ for A and B.

$\begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix} = -i - i = -2i \neq 0$, so there is a unique sol.

Thus, $\{e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t\}$ is a basis for solutions, as is $\{e^{zt}, e^{\bar{z}t}\}$.

Note: neither cos nor sin corresponds to one of z or $\bar{z}$. It's that cos, sin together correspond to $z, \bar{z}$ together. This is a common misunderstanding.

This is also obtainable from $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

In practice, e^{zt} and $e^{\bar{z}t}$ appear since they are easier to work with. Simple derivatives.

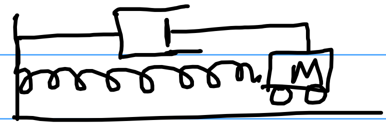
ex $y'' + y = 0.$
$r^2 + 1 = 0 \Rightarrow r = \pm i$

$y = A\cos t + B\sin t$

ex $y'' + 2y' + 4y = 0$
$r^2 + 2r + 4$
$r = \dfrac{-2 \pm \sqrt{2^2 - 4\cdot 4}}{2} = -1 \pm i\sqrt{3}$

$y = Ae^{-t}\cos(\sqrt{3}t) + Be^{-t}\sin(\sqrt{3}t)$

ex In the spring system, let's add damping which is proportional to velocity:



(that's a cylinder with fluid)

$mx'' = F = \underbrace{-kx}_{\text{spring}} \underbrace{-bx'}_{\text{damper}}$

$mx'' + bx' + kx = 0. \qquad mr^2 + br + k = 0$
$r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \dfrac{-b}{2m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - \frac{4k}{m}}$

- If $b^2 > 4mk$, then two negative roots r_1, r_2
$x = Ae^{r_1 t} + Be^{r_2 t}$. $t \to \infty$, $x \to 0$
(overdamped)

(critically damped)

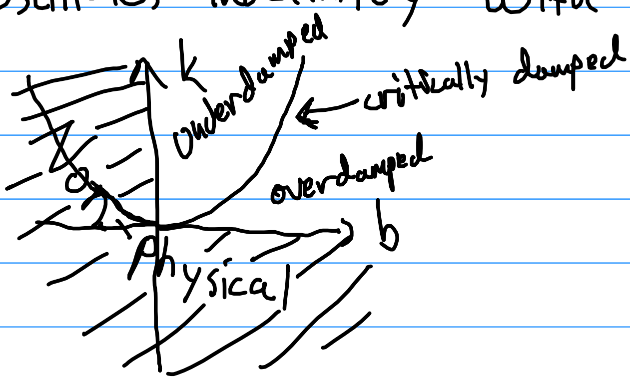
- If $b^2 = 4mk$, then one negative root.

$$x = Ae^{rt} + Bte^{rt}. \qquad t \to \infty,\ x \to 0$$

- If $b^2 < 4mk$, then let $\omega = \left(\frac{b}{m}\right)^2 - \frac{4k}{m}$

$$x = e^{-\frac{b}{2m}t}\left(A\cos\omega t + B\sin\omega t\right)$$ (underdamped)

If $b > 0$, then $x \to 0$, otherwise $b = 0$ oscillates indefinitely with same amplitude.



Method of undetermined coefficients (Guess and check)

Suppose we wish to solve $y'' + ay' + by = f(t)$, where f is some solution to some homogeneous linear differential equation. We've dealt with $f(t) = 0$ already, but what about $f(t)$ not the zero function?

First, note that if y_p is a particular solution and y_h is a homogeneous solution (to $y_h'' + ay_h' + by_h = 0$) then

$$(y_p + y_h)'' + a(y_p + y_h)' + b(y_p + y_h)$$
$$= (y_p'' + ay_p' + by_p) + (y_h'' + ay_h' + by_h) = f(t) + 0$$

so $y_p + y_h$ is another particular solution
(just like in $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{0}$)

Let's focus on finding one y_p for now.

Method:

1. identify which roots of an auxiliary polynomial $f(t)$ corresponds to
2. identify roots of auxiliary poly of $y'' + ay' + by = 0$
3. Write down general solution to homog. diff. eq. having <u>all</u> the aforementioned roots, but omit the terms which are solutions to $y'' + ay' + by = 0$.
4. Solve for coefficients so $y_p'' + ay' + by = f(t)$.

ex $y'' + 3y' + 2y = 3t$

$r^2 + 3r + 2$ $\quad$ $r = 0, 0$ (mult. two)

$= (r+2)(r+1)$

$$y_p = Ae^{0t} + Bte^{0t} = A + Bt$$

$$y_p' = B$$

$$y_p'' = 0$$

$$y_p'' + 3y_p' + 2y_p = 3B + 2A + 2Bt = 3t$$

$$\begin{cases} 2A + 3B = 0 \\ 2B = 3 \end{cases} \qquad B = 3/2, \quad A = -9/4$$

so $y_p = -\frac{9}{4} + \frac{3}{2}t$

ex $y'' + 3y' + 2y = e^{-t}$

$r = -2, -1$ $\qquad$ $r = -1$

$y_p = Ate^{-t} \cancel{+ Be^{-t} + Ce^{-2t}}$

$y_p' = -Ate^{-t} + Ae^{-t}$

$y_p'' = Ate^{-t} - Ae^{-t} - Ae^{-t}$

$(Ate^{-t} - 2e^{-t}) + 3(\quad \cdots$