July 26

Homogeneous diff.eqs

Yesterday, we learned that the solution to

$$y'' + ay' + by = 0$$

can be found by computing the roots of $r^2 + ar + b$.
But, why does this work, and is this really
all the solutions?

First observation: the solution set is a subspace.

(i) Let f, g be solutions.

$$\begin{aligned}
&(f+g)'' + a(f+g)' + b(f+g) \\
&= f'' + g'' + af' + ag' + bf + bg \\
&= (f'' + af' + bf) + (g'' + ag' + bg) \\
&= 0 + 0
\end{aligned}$$

So $f+g$ is a solution, too.

(ii) Let f be a solution, $c \in \mathbb{R}$

$$\begin{aligned}
&(cf)'' + a(cf)' + b(cf) \\
&= cf'' + caf' + cbf \\
&= c(f'' + af' + bf) \\
&= 0
\end{aligned}$$

so cf is a solution, too.

This is good because we can ask whether
there is a basis of the subspace.

Another way of seeing it is a subspace is by
using Leibniz notation:

$$\frac{d^2}{dt^2} f + a\frac{d}{dt} f + bf = 0$$

factored is $\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right) f = 0$. So the question is about $\ker\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)$, which is a subspace.

def Suppose $f_1, \cdots, f_n$ are functions $\mathbb{R} \to \mathbb{R}$. They are linearly independent if for $c_1, \cdots, c_n \in \mathbb{R}$

$\underbrace{c_1 f_1 + \cdots + c_n f_n = 0}$ only if $c_1 = \cdots = c_n = 0$.

this means $c_1 f_1(t) + \cdots + c_n f_n(t) = 0$ for all t.

ex $\sin t$ and $\cos t$ are independent.

if $c_1 \sin t + c_2 \cos t = 0$ for all t,

at $t = 0$, $\quad c_1 \cdot 0 + c_2 \cdot 1 = 0 \quad \Rightarrow c_2 = 0$

at $t = \frac{\pi}{2}$, $\quad c_1 \cdot 1 + c_2 \cdot 0 = 0 \quad \Rightarrow c_1 = 0$

so they are independent.

ex e^t and e^{2t} are independent.

if $c_1 e^t + c_2 e^{2t} = 0$ for all t,

at $t = 0$, $\quad c_1 + c_2 = 0$

at $t = \ln 2$, $\quad 2c_1 + 4c_2 = 0$

$$\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 4 - 2 \neq 0, \text{ so } c_1 = c_2 = 0 \text{ only solution.}$$

ex $1, \cos 2t, \sin^2 t$ dependent.

$1 - \cos 2t - 2\sin^2 t = 0$ is dependence.

Let's look at dependence of two functions for a moment, with an eye toward generalization.

If f, g dependent, $c_1 f + c_2 g = 0$ for c_1, c_2 not both zero. Taking the derivative of both sides, we have $c_1 f' + c_2 g' = 0$ as well. This says for all t,

$$\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0} \quad (\vec{c} \text{ nontrivial solution})$$

which is to say

$$\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = 0 \quad \text{for all } t.$$

This determinant is a <u>Wronskian</u> $W[f,g](t)$.
We have actually just shown that if $W[f,g](t)$ is ever nonzero, f,g are independent.

<u>ex</u> $f(t) = \cos t \quad g(t) = \sin t$

$$\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

so independent!

<u>Warning</u>: Look at $f(x) = x^2 \quad g(x) = x|x|$

If $c_1 x^2 + c_2 x|x| = 0$

@ $x = 1$, $\quad c_1 + c_2 = 0$

@ $x = -1$, $\quad c_1 - c_2 = 0$

$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, so $c_1 = c_2 = 0$.

So f,g independent. But:

$$g'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|$$

$$\begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2|x|x^2 - 2|x|x^2 = 0$$

Just because the Wronskian is always zero doesn't mean the functions are dependent!

<u>ex</u> If $\lambda_1 \neq \lambda_2$, $\quad e^{\lambda_1 t}, e^{\lambda_2 t}$ independent.

$$\begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{vmatrix} = \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} e^{\lambda_2 t}$$

at $t = 0$, $\quad = \lambda_2 - \lambda_1 \neq 0$.

On homework: show $e^{\lambda t}$ and $te^{\lambda t}$ independent, too.

Thus, the dimension of the solution space for a 2^{nd}-order diff. eq. is at least two. Is it at most two?

Recall: <u>thm (Existence and uniqueness)</u> For $y'' + ay' + b = 0$, $t_0, f_0, f_1 \in \mathbb{R}$, there is a solution f solving the differential equation with $f(t_0) = f_0$ and $f'(t_0) = f_1$. The solution is unique.

Why do we only need $f(t_0)$ and $f'(t_0)$? First,

$$f''(t_0) + af'(t_0) + bf(t_0) = 0 \quad \text{implies}$$

$$f''(t_0) = -af'(t_0) - bf(t_0).$$

Taking the derivative of the diff-eq., $y''' + ay'' + by' = 0$ implies $f'''(t_0) = -af''(t_0) - bf'(t_0)$, so $f'''(t_0)$ fixed, too. Taking more and more derivatives,

$$f^{(n+2)}(t_0) = -af^{(n+1)}(t_0) - bf^{(n)}(t_0)$$

Using $f_k = f^{(k)}(t_0)$, we have $f_{k+2} = -af_{n+1} - bf_n$. This is a linear recurrence! All derivatives of all orders are determined by only f_0 and f_1!

In fact, then

$$f(t) = \sum_{n=0}^{\infty} \frac{f_n}{n!}(t-t_0)^n$$

where the series is determined by only the first two terms.

The theorem actually gives an isomorphism

$$\text{Solutions to diff. eq.} \longrightarrow \mathbb{R}^2$$
$$f \longmapsto \begin{bmatrix} f(t_0) \\ f'(t_0) \end{bmatrix}$$

"existence" means this transformation is onto
"uniqueness" means it is one-to-one.

Let's check it is a linear transformation.

(i) Suppose f, g solutions. $(f+g)(t_0) = f(t_0) + g(t_0)$
and $(f+g)'(t_0) = f'(t_0) + g'(t_0)$

so $f+g$ maps to $\begin{bmatrix} f(t_0) \\ f'(t_0) \end{bmatrix} + \begin{bmatrix} g(t_0) \\ g'(t_0) \end{bmatrix}$.

(ii) Suppose f a solution, $c \in \mathbb{R}$. $(cf)(t_0) = cf(t_0)$
and $(cf)'(t_0) = cf'(t_0)$

so cf maps to $c\begin{bmatrix} f(t_0) \\ f'(t_0) \end{bmatrix}$.

We have an isomorphism! Thus, the dimension of the space of solutions to a 2^{nd}-order linear homogeneous diff. eq. is exactly 2!

We can use this to extend the Wronskian stuff: if f, g solutions to $y'' + ay' + by = 0$ and there is some $t_0 \in \mathbb{R}$ where $W[f,g](t_0) = 0$, then f, g dependent.

$$W[f,g](t_0) = \begin{vmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{vmatrix}, \quad \text{so} \quad \begin{bmatrix} f(t_0) \\ f'(t_0) \end{bmatrix}, \begin{bmatrix} g(t_0) \\ g'(t_0) \end{bmatrix}$$

are dependent vectors. By the isomorphism, f, g are dependent functions. (Isomorphisms bring independent vectors to independent vectors. This is why they preserve dimension.)

Then, since dependent, $W[f,g](t)$ always 0.

The contrapositive is that if f, g independent, not only is $W[f,g](t)$ nonzero somewhere, but everywhere.
Warning: only true if f, g solns to 2nd order homog. lin. diff. eq.!

Why the auxiliary equation? We know e^{rt} is a solution to $y'' + ay' + by = 0$

$$(e^{rt})'' + a(e^{rt})' + be^{rt} = r^2 e^{rt} + rae^{rt} + be^{rt}$$

and since $e^{rt} \neq 0$ ever, we just need

$$r^2 + ar + b = 0$$

for e^{rt} to be a solution. Double roots: all you need to do is show te^{rt} is a solution, it is independent of e^{rt}, so it is solved!

More advanced: find $\ker\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)$

Let λ_1, λ_2 be roots of $r^2 + ar + b$. Then,

$$\frac{d^2}{dt^2} + a\frac{d}{dt} + b = \left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} - \lambda_2\right).$$

Then, $\ker\left(\frac{d}{dt} - \lambda_2\right) \subset \ker\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)$.

What is $\ker\left(\frac{d}{dt}-\lambda_2\right)$? It is all f where
$\left(\frac{d}{dt}-\lambda_2\right)f=0$. That is, $\frac{d}{dt}f=\lambda_2 f$.
Separation of variables $\Longrightarrow$ $f=Ae^{\lambda_2 t}$ for
$A\in\mathbb{R}$ free !

If $\lambda_1\neq\lambda_2$, then we obtain $e^{\lambda_1 t}, e^{\lambda_2 t}$, which are indep., so by dim. we are done.

If $\lambda_1=\lambda_2$, then we have

$$\left(\frac{d}{dt}-\lambda_1\right)\left(\frac{d}{dt}-\lambda_1\right)f=0$$

So another solution comes from

$$\left(\frac{d}{dt}-\lambda_1\right)f=\underbrace{e^{\lambda_1 t}}_{\text{killed by next } \frac{d}{dt}-\lambda_1}$$

This is solvable by an integrating factor, which gives $f=Bte^{\lambda_1 t}$. We have $e^{\lambda_1 t}, te^{\lambda_1 t}$, which are indep., so done by dimension.