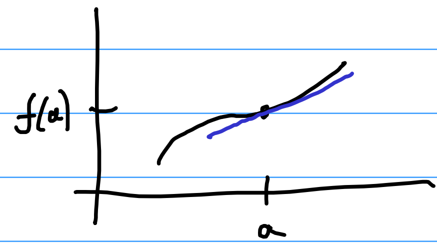
Differential Equations

July 25

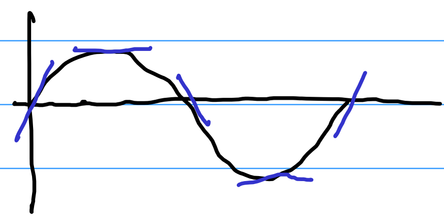
Recall: the derivative of a function $f: \mathbb{R} \to \mathbb{R}$ is a function $f': \mathbb{R} \to \mathbb{R}$ where $f'(a)$ is the slope of the tangent line at $(a, f(a))$

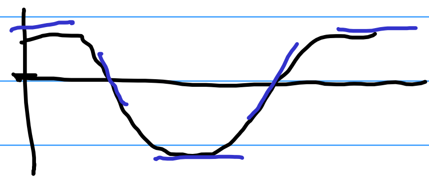


If a function has a derivative, it is differentiable.

ex For x a variable, $c \in \mathbb{R}$, n an integer >0,

$$c' = 0 \qquad (e^{cx})' = ce^{cx}$$
$$(x^n)' = nx^{n-1} \qquad (\cos x)' = -\sin x$$
$$(\sin x)' = \cos x$$





nonex $|x|$ is not differentiable at 0.

A second derivative $f'': \mathbb{R} \to \mathbb{R}$ is the derivative of the derivative of f. Again, these might not exist. For instance, $f(x) = x|x|$ is differentiable but not second-differentiable.

The n^{th} derivative $f^{(n)}: \mathbb{R} \to \mathbb{R}$ is the derivative of $f^{(n-1)}: \mathbb{R} \to \mathbb{R}$.

A differential equation is an equation involving an unknown function and its derivatives.

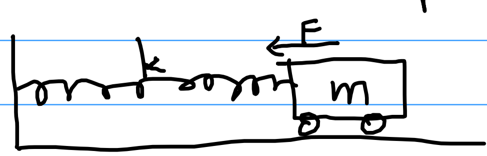
ex $f' = 2f$ $\qquad f'' - f' + f = 0$

$$xg'(x) = \cos x$$

A solution is a function (which is differentiable enough) which satisfies the diff. eq. when substituted in.

ex $f(x) = e^{2x}$ is a solution to $f' = 2f$.
since $f' = 2e^{2x} = 2f$.

Differential equations come up when modeling continuous physical systems. The classic example is a mass on a spring:



Let x be the position of the cart, relative to rest length, and x' its velocity. Newton's 2nd law is the differential equation $F = mx''$. x'' is the acceleration of the cart, F the force applied to it. Hooke's law for springs is

$$F = -kx$$

So, $mx'' = -kx$ is the differential equation governing its motion.

As a guess, guided by seeing the second derivative is negative itself, perhaps $x = A\cos(\omega t)$

$$x' = -A\omega \sin(\omega t) \qquad x'' = -A\omega^2 \cos(\omega t)$$

so

$$mx'' = -mA\omega^2 \cos(\omega t)$$
$$-kx = -kA\cos(\omega t)$$

which are equal if $m\omega^2 = k$.

$$x = A\cos\left(\sqrt{\tfrac{k}{m}}\, t\right) \text{ is a solution.}$$

This is periodic motion, as one would expect. k decrease or m increase gives slower motion.

Plan:

- Study linear second-order diff. eqs } warmup for
- study linear higher-order diff. eqs }
- study differential equations

$$\vec{x}'(t) = A\vec{x}(t) \qquad \vec{x} : \mathbb{R} \to \mathbb{R}^n$$
A $n \times n$ matrix.

Many physical systems can be described or approximated by linear diff. eqs. The above cart system could instead be written as

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

This is especially useful when there are multiple interacting objects with their own positions and velocities!

Linear 2nd-order diff. eqs

A linear 2nd-order diff. eq is a diff. eq. of the form $f'' + af' + bf = g$, where $a, b \in \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ some function, and f unknown.
Could also write $f''(t) + af'(t) + bf(t) = g(t)$.
g is sometimes interpreted as a driving term.

A homogeneous linear 2nd-order diff. eq. is one where $g(t) = 0$. So: $f'' + af' + bf = 0$.

Let's start by giving the way to solve them:

1. The auxiliary polynomial is $r^2 + ar + b$.
   Let λ_1, λ_2 be the roots.

2. • if $\lambda_1 \neq \lambda_2$, $f(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ is a solution for all $A, B \in \mathbb{R}$.
   • if $\lambda_1 = \lambda_2$, $f(t) = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}$ instead

ex $f'' - 3f' + 2f = 0$.

$$r^2 - 3r + 2 = (r-2)(r-1)$$

$$f(t) = Ae^{2t} + Be^{t}$$

ex $f'' - 4f' + 4f = 0$

$$r^2 - 4r + 4 = (r-2)^2$$

$$f(t) = Ae^{2t} + Bte^{2t}.$$

If the roots are complex, it actually still works.

ex
$$f'' + f = 0$$
$$r^2 + 1 = 0$$
$$r = \pm i$$

$$f(t) = Ae^{it} + Be^{-it}$$

Since
$$e^{it} = \cos t + i\sin t$$
$$e^{-it} = \cos t - i\sin t$$

$$f(t) = (A+B)\cos t + (A-B)i\sin t$$

It's possible to solve
$$A+B = C$$
$$Ai - Bi = D$$
for any C, D, so
$$f(t) = C\cos t + D\sin t.$$

In general, if $\lambda = a+bi$ is a root, so is $\bar{\lambda} = a - bi$, and together these give
$$f(t) = e^{at}(A\cos(bt) + B\sin(bt))$$
which is useful if you like sine and cosine.

Thm Existence and uniqueness: For $f'' + af' + bf = 0$ $(a, b \in \mathbb{R})$ and $t_0 \in \mathbb{R}$, $f_0, f_1 \in \mathbb{R}$, there is a unique solution f with $f(t_0) = f_0$ and $f'(t_0) = f_1$.

ex $f'' - 3f' + 2f = 0, \quad f(0) = 1, \quad f'(0) = 2$

$f(t) = Ae^{2t} + Be^{t} \qquad f(0) = A + B = 1$

$f'(t) = 2Ae^{2t} + Be^{t} \qquad f'(0) = 2A + B = 2$

solvable.