July 21

Spectral theorem

A symmetric matrix is an $n \times n$ matrix A with $A = A^T$.
They arise as matrices of inner products on $\mathbb{R}^n$, and more generally, "symmetric bilinear forms" $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$.

ex $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ nonex $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

("skew-symmetric")

Thm If A symmetric and $\vec{v}, \vec{w}$ are vectors in two distinct eigenspaces, $\vec{v}$ and $\vec{w}$ are orthogonal.

pf Suppose $A\vec{v} = \lambda \vec{v}$, $A\vec{w} = \mu \vec{w}$, and $\lambda \neq \mu$.

$\vec{v} \cdot A\vec{w} = \vec{v} \cdot \mu \vec{w} = \mu(\vec{v} \cdot \vec{w})$

" $\vec{v}^T A \vec{w} = \vec{v}^T A^T \vec{w} = (A\vec{v}) \cdot \vec{w} = \lambda \vec{v} \cdot \vec{w}$

so, $(\mu - \lambda)(\vec{v} \cdot \vec{w}) = 0$. Since $\mu - \lambda \neq 0$, $\vec{v} \cdot \vec{w} = 0$.

Notice the general fact about symmetric matrices:

$$\vec{v} \cdot A\vec{w} = A\vec{v} \cdot \vec{w}$$

A matrix is orthogonally diagonalizable if it can be diagonalized as $A = PDP^{-1}$ with P an orthogonal matrix. So, $A = PDP^T$ is equivalent.

If $A = PDP^T$, $A^T = (PDP^T)^T = PDP^T$, so orthogonally diagonalizable matrices are symmetric.

Suppose a symmetric matrix is diagonalizable. We have different eigenspaces being orthogonal to each other, so by doing Gram-Schmidt on each eigenspace, we can get an orthonormal basis of eigenvectors, and hence an orthonormal P matrix.

ex $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((2-\lambda)(1-\lambda) - (\lambda-1)\big) - \big((1-\lambda) - (\lambda - 1)\big)$$

$$= (2-\lambda)(3-\lambda)(1-\lambda) + 2\lambda$$

$\ldots \quad \lambda = 1, 1, 4.$

$$\mathrm{Nul}(A - 4I) = \mathrm{Nul}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ e.vector $\quad \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ normalized

$\mathrm{Nul}(A - I) = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ eigenvectors.

Gram-Schmidt: $\dfrac{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} = \dfrac{1}{2} \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$

orthonormal basis of eigenspace: $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

So $A = PDP^T$ with $P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}$

$$D = \begin{pmatrix} 4 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

It isn't a coincidence that this can always be done.

Spectral theorem for diagonal matrices

If $n \times n$ A is symmetric, then
1. A has n real eigenvalues, with multiplicity
2. If λ is an eigenvalue, the multiplicity of λ is the dimension of eigenspace λ.
3. The eigenspaces are mutually orthogonal.
4. A is orthogonally diagonalizable.

partial proof:

for 1, suppose λ is an eigenvalue, $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \in \mathbb{C}^n$. (!)

In $\mathbb{C}$, $\overline{\vec{v}^T}\,\vec{v}$ is length.

$$\overline{\vec{v}^T} A\vec{v} = \overline{\vec{v}^T}\lambda\vec{v} = \lambda\,\overline{\vec{v}^T}\vec{v}$$

$$" \; \overline{\vec{v}^T}\,\overline{A^T}\vec{v} = \overline{(A\vec{v})^T}\,\vec{v} = \overline{(\lambda\vec{v})^T}\,\vec{v} = \overline{\lambda}\,\overline{\vec{v}^T}\vec{v}$$

A has real entries

So $\lambda = \overline{\lambda}$. This means λ is real.

for 4, Ch 6 supp. ex. 16 describes Schur Factorization:
if A is $n \times n$ with n real eigenvalues (with mult.), there is orthogonal U and upper triangular R with $A = URU^T$.

Since $A^T = UR^TU^T$, then $R = R^T$, so is diagonal.
This is an orthogonal diagonalization!

for 3, we've already spoken about it

for 2, it is because A is diagonalized. ∎

Spectral decomposition

Given $A = PDP^T$ an orthogonal diagonalization,

$$A = \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} — \vec{u}_1 — \\ \vdots \\ — \vec{u}_n — \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \vec{u}_1 & \cdots & \lambda_n \vec{u}_n \end{pmatrix} \begin{pmatrix} — \vec{u}_1 — \\ \vdots \\ — \vec{u}_n — \end{pmatrix}$$

$$= \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

is the *spectral decomposition* of A. Notice $\vec{u}_i \vec{u}_i^T$ is the matrix of $\text{proj}_{\vec{u}_i}$, so

$$A\vec{v} = \lambda_1 \text{proj}_{\vec{u}_1} \vec{v} + \cdots + \lambda_n \text{proj}_{\vec{u}_n} \vec{v}.$$

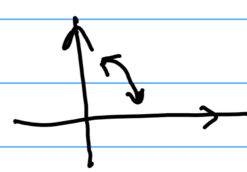
ex If $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \vec{u}_1 \vec{u}_1^T - \vec{u}_2 \vec{u}_2^T.$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



For inner products, $\langle \vec{x}, \vec{y} \rangle_A = \vec{x}^T A \vec{y}$, then since $A = PDP^T$, $= \vec{x}^T PDP^T \vec{y} = (P^T\vec{x})^T D (P^T\vec{y}) = \langle P^T\vec{x}, P^T\vec{y} \rangle_D$.
Since P is like a rotation, this is saying every inner product can be "rotated" into an inner product with a diagonal matrix.
Positive definiteness implies the entries on the diagonal of D are positive (so eigenvalues are positive)

Schur factorization

Say A $n \times n$ with n real eigenvalues. Want orthogonal U and upper triangular R with $A = URU^T$.

1. If $n=1$, then $A = I_1 A I_1^T$ is a Schur factorization.
2. If $n>1$ and Schur factorization works for $(n-1)\times(n-1)$ matrices:

   Since A has an eigenvalue λ let $\vec{v}$ be an eigenvector. Let $\vec{u}_1 = \frac{1}{\|\vec{v}\|}\vec{v}$. Let $\vec{u}_2, \dots, \vec{u}_n$ complete an orthonormal basis of $\mathbb{R}^n$. $U = (\vec{u}_1 \cdots \vec{u}_n)$

   The matrix of A in this basis looks like

$$\left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right)$$

   $|A - \lambda I| = (\lambda_1 - \lambda)|A' - \lambda I|$, so A' has $n-1$ real eigenvalues.

   $A' = P'R'(P')^T$ Schur factorization (A' is $(n-1)\times(n-1)$)

$$A = U\left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & P'R'(P')^T & \\ 0 & & & \end{array}\right) = U\left(\begin{array}{c|c} 1 & \\ \hline & P' \end{array}\right)\left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & R' & \\ 0 & & & \end{array}\right)\left(\begin{array}{c|c} 1 & \\ \hline & P' \end{array}\right)^T U^T.$$