July 18

Last time:

- For subspace $W < \mathbb{R}^n$, $\dim W + \dim W^\perp = n$
  (Goal: for $\vec{v} \in \mathbb{R}^n$, write $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$ with $\vec{v}^{\parallel} \in W$ and $\vec{v}^{\perp} \in W^\perp$)
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ an orthogonal set if $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$
- Orthogonal sets of nonzero vectors are independent.
  (Goal: produce orthogonal bases of a vector space)

Suppose $\vec{u}_1, \dots, \vec{u}_p$ is an orthogonal basis of $W < \mathbb{R}^n$, and let $\vec{v} \in W$. What are the coordinates of $\vec{v}$?

$$\vec{v} = c_1\vec{u}_1 + \dots + c_p\vec{u}_p \quad \text{for some } \vec{c} \in \mathbb{R}^p$$

$$\text{so } \vec{u}_i \cdot \vec{v} = c_1(\vec{u}_i \cdot \vec{u}_1) + \dots + c_i(\vec{u}_i \cdot \vec{u}_i) + \dots + c_p(\vec{u}_i \cdot \vec{u}_p)$$
$$= 0 + \dots + c_i(\vec{u}_i \cdot \vec{u}_i) + \dots + 0$$

so $c_i = \dfrac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i}$. No need to solve a system of equations! Note c_i is constant in $\mathrm{proj}_{\vec{u}_i} \vec{v}$.

An orthonormal set is an orthogonal set of unit-length vectors. (Thus, nonzero, so they are independent sets.)

ex $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \right\}$

The Kronecker delta, is $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$. In other words, $(\delta_{ij})_{ij} = I_n$.

Thm An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

pf Suppose $U = (\vec{u}_1 \ \cdots \ \vec{u}_n)$. $U^T U = \begin{pmatrix} - \vec{u}_1 - \\ \vdots \\ - \vec{u}_n - \end{pmatrix} (\vec{u}_1 \ \cdots \ \vec{u}_n)$

$$= (\vec{u}_i \cdot \vec{u}_j)_{ij} = (\delta_{ij})_{ij} = I_n.$$

Properties For $m\times n$ U with orthonormal columns,

(1) $U\vec{x}\cdot U\vec{y} = \vec{x}\cdot\vec{y}$ (left side in $\mathbb{R}^m$,

(2) $\|U\vec{x}\| = \|\vec{x}\|$ right side in $\mathbb{R}^n$)

pf (1) $U\vec{x}\cdot U\vec{y} = (U\vec{x})^T(U\vec{y}) = \vec{x}^T U^T U\vec{y} = \vec{x}^T I_n \vec{y} = \vec{x}\cdot\vec{y}$

(2) $\|U\vec{x}\| = \sqrt{U\vec{x}\cdot U\vec{x}} = \sqrt{\vec{x}\cdot\vec{x}} = \|\vec{x}\|$.

These say that the map $\vec{x}\mapsto U\vec{x}$ is angle- and length-preserving.

An <u>orthogonal matrix</u> (sometimes <u>orthonormal matrix</u>) is a square matrix with orthonormal columns. This is basically the definition: $U^TU = I_n$ and U is $n\times n$.

Thus, $U^{-1} = U^T$. Since $UU^{-1} = I_n$, too, and $(U^{-1})^T = U$, then U^{-1} is also orthonormal!

ex $U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ is orthonormal, so U^T is its inverse.

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthonormal, so $R_\theta^T = U^T = R_{-\theta}$ is its inverse.

ex If U $n\times n$ orthogonal,

$$\det(U^TU) = \det(I_n)$$

$$\det(U)^2 = 1, \quad \text{so} \quad \det(U) = \pm 1$$

If n is odd, U has a real eigenvalue λ. (nonzero)

$U\vec{v} = \lambda\vec{v}$ so $\|U\vec{v}\| = \|\lambda\vec{v}\|$

$\|\vec{v}\| = |\lambda|\,\|\vec{v}\|$ so $\lambda = \pm 1$

Orthogonal projection

Given $W \subset \mathbb{R}^n$ a subspace, and $\vec{v} \in \mathbb{R}^n$, want

$\vec{v}^{\parallel} \in W$ and $\vec{v}^{\perp} \in W^{\perp}$ so that $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$

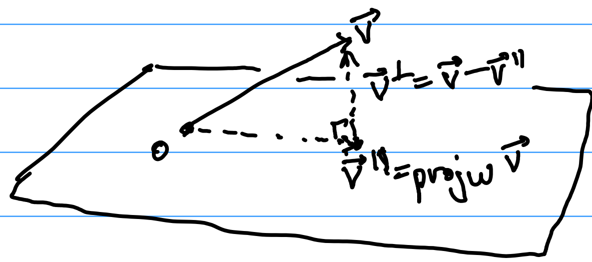
Suppose W has orthogonal basis $\vec{u}_1, \dots, \vec{u}_p$.

Let $\vec{v}^{\parallel} = c_1 \vec{u}_1 + \cdots + c_p \vec{u}_p$ with $c_i = \dfrac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i}$

$$\vec{u}_i \cdot (\vec{v} - \vec{v}^{\parallel}) = \vec{u}_i \cdot \vec{v} - \vec{u}_i \cdot v^{\parallel} = \vec{u}_i \cdot \vec{v} - \frac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i} \vec{u}_i \cdot \vec{u}_i = 0.$$

Since this is true, $\vec{v} - \vec{v}^{\parallel} \in W^{\perp}$. Let $\vec{v}^{\perp} = \vec{v} - \vec{v}^{\parallel}$.

Hence, $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$, as required.



Theoretical result: $\mathbb{R}^n$ isomorphic to $W \oplus W^{\perp}$ by

$\vec{v} \longmapsto (\text{proj}_W \vec{v}, \vec{v} - \text{proj}_W \vec{v})$.

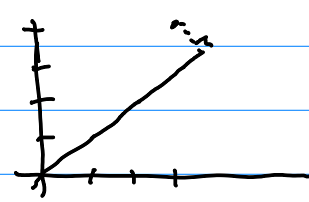
$\oplus$ is "direct sum". Pairs as a vector space.

Inverse is $(\vec{w}^{\parallel}, \vec{w}^{\perp}) \longmapsto \vec{w}^{\parallel} + \vec{w}^{\perp}$.

So, $\dim \mathbb{R}^n = \dim (W \oplus W^{\perp}) = \dim W + \dim W^{\perp}$.

ex $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

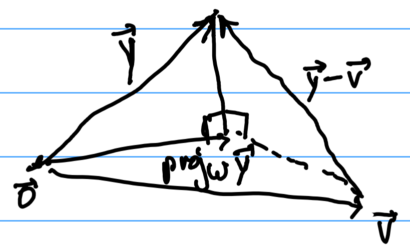
$$\vec{v}^{\parallel} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \vec{v}^{\perp} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$



If $\vec{v} \in W$, $\text{proj}_W \vec{v} = \vec{v}$, using coordinates result from earlier.

Thm For $W \subset \mathbb{R}^n$ subspace and $\vec{y} \in \mathbb{R}^n$, for any $\vec{v} \in W$, $\|\vec{y} - \text{proj}_W \vec{y}\| \leq \|\vec{y} - \vec{v}\|$. That is, $\text{proj}_W \vec{y}$ is the closest vector in W to $\vec{y}$.

pf



$$\|\vec{y}-\vec{v}\|^2 = \|\vec{y} - \text{proj}_W \vec{y}\|^2 + \|\vec{v} - \text{proj}_W \vec{y}\|^2$$
$$\geq \|y - \text{proj}_W \vec{y}\|^2$$

so $\|\vec{y} - \vec{v}\| \geq \|y - \text{proj}_W \vec{y}\|$.

Thm If $W = \text{Col}\, U$ for U an $m \times n$ matrix with orthonormal columns, $\text{proj}_W \vec{v} = UU^T\vec{v}$.

pf $U = (\vec{u}_1 \cdots \vec{u}_n)$. $\text{proj}_W \vec{v} = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{v})\vec{u}_n$, since $\vec{u}_i \cdot \vec{u}_i = 1$. so $= (\vec{u}_1 \cdots \vec{u}_n)\begin{pmatrix} \vec{u}_1 \cdot \vec{v} \\ \vdots \\ \vec{u}_n \cdot \vec{v} \end{pmatrix}$.

Since $\begin{pmatrix} \vec{u}_1 \cdot \vec{v} \\ \vdots \\ \vec{u}_n \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} u_1^T \vec{v} \\ \vdots \\ u_n^T \vec{v} \end{pmatrix} = \begin{pmatrix} - \vec{u}_1 - \\ \vdots \\ - \vec{u}_n - \end{pmatrix} \vec{v} = U^T \vec{v}$, we have $\text{proj}_W \vec{v} = UU^T\vec{v}$.

ex For $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\text{proj}_{\vec{v}} \vec{x} = \vec{v}v^T\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1)\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\vec{x}$.

So, (1) $U^TU = I_n$

(2) $UU^T = \text{proj}_W$

When U is square, $\text{Col}\, U = \mathbb{R}^n$, so $\text{proj}_W = I_n$, too.

Gram-Schmidt

Given a basis $\vec{v}_1, \ldots, \vec{v}_p$ of a subspace W, we can obtain an orthogonal basis of W step-by-step from the given basis.

Defining rules

1. After step n, we have replaced $\vec{v}_1, \cdots, \vec{v}_n$ with an orthogonal set $\vec{u}_1, \cdots, \vec{u}_n$
2. $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \text{Span}\{\vec{u}_1, \cdots, \vec{u}_n\}$

Thus, after step p, we have an orthogonal basis.
To implement these rules, simply:

$$\vec{u}_n = \vec{v}_n - \text{proj}_{W_n} \vec{v}_n, \text{ where } W_n = \text{Span}\{\vec{u}_1, \cdots, \vec{u}_{n-1}\}$$

Thus, $\vec{u}_n$ is in $W_n^\perp$, so is orthogonal to $u_1, \cdots, u_{n-1}$.

Then, if $\text{Span}\{\vec{u}_1, \cdots, \vec{u}_{n-1}\} = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_{n-1}\}$,

$$\begin{aligned}\text{Span}\{\vec{u}_1, \cdots, \vec{u}_n\} &= \text{Span}\{\vec{u}_1, \cdots, \vec{u}_{n-1}, \vec{v}_n - \text{proj}_{W_n}\vec{v}_n\} \\ &= \text{Span}\{\vec{u}_1, \cdots, \vec{u}_{n-1}, \vec{v}_n\} \\ &= \text{Span}\{\vec{v}_1 \cdots, \vec{v}_{n-1}, \vec{v}_n\}.\end{aligned}$$

ex $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ of $\mathbb{R}^3$.

$\vec{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. $\dfrac{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}} = \frac{1}{3}$, so $\vec{u}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix}$

$\dfrac{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}} = \frac{1}{3}$ and $\dfrac{\begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\vec{u}_2 \cdot \vec{u}_2} = \dfrac{-1/3}{6/9} = -\frac{1}{2}$, $\vec{u}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix} = \begin{bmatrix}0\\1/2\\-1/2\end{bmatrix}$

$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix}, \begin{bmatrix}0\\1/2\\-1/2\end{bmatrix}$ is orthogonal basis by Gram-Schmidt.

Obtain an orthonormal basis by normalizing.