Jul 12

to review:

- an <u>eigenvector</u> $\vec{v} \in \mathbb{R}^n$ with <u>eigenvalue</u> $\lambda \in \mathbb{R}$ for $n \times n$ A is a nonzero vector satisfying $A\vec{v} = \lambda \vec{v}$ (A just scales it)

  ex $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- the <u>eigenspace</u> for λ is $\mathrm{Nul}(A - \lambda I)$; eigenvectors are nonzero vectors in this subspace. Principal problem: find bases of eigenspaces.
- the <u>characteristic polynomial</u> for A is $p_A(\lambda) = |A - \lambda I|$. Its roots are eigenvalues of A.
- the $\lambda = 0$ eigenspace is just $\mathrm{Nul}(A)$ — so eigenvalues are a generalization (also: A invertible $\iff$ 0 is not an eigenvalue of A)

Similarity

Let A, B be $n \times n$. A is <u>similar</u> to B if there is an invertible $n \times n$ P with $A = PBP^{-1}$ (or: $AP = PB$).

$$\begin{tikzcd}
\vec{x} \in \mathbb{R}^n \arrow[r, "B"] \arrow[d, "P"] & \mathbb{R}^n \ni B\vec{x} \arrow[d, "P"] \\
P\vec{x} \in \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \ni PB\vec{x} = AP\vec{x}
\end{tikzcd}$$

if B is the matrix of A relative to a basis P, then they are similar.

- A matrix is similar to itself: $A = I_n A I_n^{-1}$
- If A sim. to B, B sim. to A: $B = (P^{-1}) A (P^{-1})^{-1}$
- If A sim. to B sim. to C, A sim. to C:

  $A = PBP^{-1}$, $B = QCQ^{-1}$: $A = (PQ)C(PQ)^{-1}$.

Thm If A similar to B, they have the same characteristic polynomial.

pf $|A-\lambda I_n| = |PBP^{-1} - \lambda I_n|$
$= |PBP^{-1} - \lambda PP^{-1}|$
$= |P(B-\lambda I_n)P^{-1}|$
$= |P|\,|B-\lambda I_n|\,|P^{-1}|$
$= |B-\lambda I_n|\,|PP^{-1}|$
$= |B-\lambda I_n|.$

Thm If A, B similar, then $\dim \mathrm{Nul}(A-\lambda I) = \dim \mathrm{Nul}(B-\lambda I)$.

pf If $A\vec{v} = \lambda\vec{v}$, then $PBP^{-1}\vec{v} = \lambda\vec{v}$, so $BP^{-1} = \lambda P^{-1}\vec{v}$.
This means $P^{-1}\vec{v}$ is an eigenvector of B, with same eigenvalue.
So, $P^{-1}: \mathrm{Nul}(A-\lambda I) \longrightarrow \mathrm{Nul}(B-\lambda I)$ is a linear transformation
and $\dim \mathrm{Nul}(A-\lambda I) \leq \dim \mathrm{Nul}(B-\lambda I)$ since P^{-1} is one-to-one.
Swapping roles of A, B gets us equality.
That is, $\vec{v}_1, \ldots, \vec{v}_k$ independent $\Rightarrow P^{-1}\vec{v}_1, \ldots, P^{-1}\vec{v}_k$ indep., too.

ex $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ have same char. poly $(1-\lambda)^2$
but are not similar (2-dim vs 1-dim e.spaces)

ex Matrices similar to $\begin{pmatrix} c & \\ & c \end{pmatrix}$? $(c \in \mathbb{R})$

$P\begin{pmatrix} c & \\ & c \end{pmatrix}P^{-1} = P(cI_n)P^{-1} = cI_n$. So, only $\begin{pmatrix} c & \\ & c \end{pmatrix}$.

Diagonalization

If a matrix is similar to a diagonal matrix, it is **diagonalizable**. $A = PDP^{-1}$ with D diagonal is a **diagonalization**.

A dynamical system describes a point/state through time.
One kind is $\vec{x}_{n+1} = A\vec{x}_n$, with n the nth time step.

So $\vec{x}_1 = A\vec{x}_0$, $\vec{x}_2 = A\vec{x}_1 = A^2\vec{x}_0$, ---, $\vec{x}_n = A^n\vec{x}_0$.

If A is diagonalizable, $A = PDP^{-1}$, we have

$$\begin{aligned} \vec{x}_n &= (PDP^{-1})^n \vec{x} \\ &= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})\vec{x} \\ &= PD(P^{-1}P)D(P^{-1}P)D \cdots DP^{-1}\vec{x} \\ &= PD^nP^{-1}\vec{x}_0. \end{aligned}$$

How has this helped?

$$DD = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_k^2 \end{pmatrix}$$

in fact,

$$D^n = \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{pmatrix}$$ Very easy to calculate!

Question: When is a matrix diagonalizable?

Observation 1: Let $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$A(P\vec{e}_i) = PDP^{-1}P\vec{e}_i = PD\vec{e}_i = P(\lambda_i\vec{e}_i) = \lambda_i(P\vec{e}_i)$$

So: the columns of P are eigenvectors of A, with the diagonal entries of D the corresponding eigenvalues.

Observation 2: Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis of $\mathbb{R}^n$ which are eigenvectors of A, $\lambda_1, \dots, \lambda_n$ the corres. eigenvals.

$$A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n$$

So, relative basis $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$,

A is a diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Thm $n\times n$ matrix A is diagonalizable if and only if there is a basis of $\mathbb{R}^n$ of eigenvectors of $\mathbb{R}^n$ (i.e., n lin. indep. eigenvectors of A)

This suggests the following diagonalization method:

1. find eigenvalues (roots of $|A-\lambda I|$)
2. find bases of eigenspaces
3. if have enough, let P have as columns the vectors from all the bases, and D the corr. eigenvals.

In fact, we do not always need to do step 2 if testing diagonalizability is all we care about:

Thm If A $(n\times n)$ has n distinct eigenvalues, it is diagonalizable.

Pf Each eigenspace has at least one eigenvector, so we get n eigenvectors with distinct eigenvalues. By yesterday's theorem, they are independent. Thus, A diagonalizable.

ex Diagonalize $\begin{pmatrix}1&2\\2&1\end{pmatrix}$.

$$|A-\lambda I| = (1-\lambda)^2 - 4 = (\lambda-3)(\lambda+1)$$
$$\lambda = 3, -1$$

Hence it is diagonalizable.

$\lambda=3$: $\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda=-1$: $\begin{bmatrix}1\\-1\end{bmatrix}$ (from yesterday)

$$\begin{pmatrix}1&2\\2&1\end{pmatrix} = \begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}3&\\&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}.$$

diagonalized!

ex Of course, don't need nondistinct:

$A = I_2$ has $\lambda = 1, 1$

$A = I_2 I_2 I_2^{-1}$

Though, repeated eigenvalues allow nondiagonalizable!

$A = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ has $\lambda = 1, 1$, but $Nul(A-\lambda I) = Span\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$,

so not enough eigenvectors to form a basis!

(however, $A^n = \begin{pmatrix}1&n\\0&1\end{pmatrix}$, which is easy enough)

Aside: Jordan Normal Form is the next-best thing to diagonalization. Every matrix is similar to a Jordan matrix — theoretically useful!

ex The Fibonacci sequence goes $0, 1, 1, 2, 3, 5, 8, \ldots$ with $x_{n+2} = x_n + x_{n+1}$. Let $\vec{x}_n = \begin{bmatrix}x_{n-1}\\x_n\end{bmatrix}$ be the "state vector".

$$\vec{x}_{n+1} = \begin{bmatrix}x_n\\x_{n+1}\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}x_{n-1}\\x_n\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\vec{x}_n$$

For $\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x}_{n+1} = A^n \vec{x}_1$ gives the Fibonacci sequence!

$$|A - \lambda I_2| = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Let $\phi = \frac{1+\sqrt{5}}{2}$ (the "golden ratio")

The other root is $1 - \phi = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$.

Two distinct eigenvalues implies both eigenspaces are 1-dim.

$\lambda = \phi$: $Nul(A - \phi I) = Nul\begin{pmatrix} -\phi & 1 \\ 1 & 1-\phi \end{pmatrix}$

trick: must have exactly 1 pivot, so second row is multiple of first.

$$= Nul\begin{pmatrix} -\phi & 1 \\ 0 & 0 \end{pmatrix} = Span\left\{\begin{bmatrix} 1 \\ \phi \end{bmatrix}\right\}$$

$\lambda = 1 - \phi$: $Nul(A - (1-\phi)I) = Nul\begin{pmatrix} -(1-\phi) & 1 \\ 1 & \phi \end{pmatrix}$

again, $= Nul\begin{pmatrix} -(1-\phi) & 1 \\ 0 & 0 \end{pmatrix} = Span\left\{\begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}\right\}$

So $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ \phi & 1-\phi \end{bmatrix}$ $D = \begin{bmatrix} \phi & \\ & 1-\phi \end{bmatrix}$

Preliminary: $P^{-1} = \frac{1}{1-\phi-\phi}\begin{bmatrix} 1-\phi & -1 \\ -\phi & 1 \end{bmatrix}$ and $\frac{1}{1-2\phi} = \frac{-1}{\sqrt{5}}$

$$\vec{x}_{n+1} = \begin{bmatrix} 1 & 1 \\ \phi & 1-\phi \end{bmatrix}\begin{bmatrix} \phi^n & \\ & (1-\phi)^n \end{bmatrix}\frac{-1}{\sqrt{5}}\begin{bmatrix} 1-\phi & -1 \\ -\phi & 1 \end{bmatrix}\vec{x}_1$$

Since $\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \phi & 1-\phi \end{bmatrix} \begin{bmatrix} \phi^n & \\ & (1-\phi)^n \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ \phi & 1-\phi \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \phi^n \\ \frac{-1}{\sqrt{5}} (1-\phi)^n \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} (1-\phi)^n \\ \frac{1}{\sqrt{5}} \phi^{n+1} - \frac{-1}{\sqrt{5}} (1-\phi)^{n+1} \end{bmatrix}$$

Thus, $x_n = \frac{1}{\sqrt{5}} \left(\phi^n - (1-\phi)^n \right)$!

Since $1-\phi \approx -0.618$, $\frac{1}{\sqrt{5}}(1-\phi)^n < \frac{1}{2}$ for all $n \geq 0$,

so $x_n \approx \frac{1}{\sqrt{5}} \phi^n$ (if fact, x_n is $\frac{1}{\sqrt{5}} \phi^n$ rounded to the nearest integer!)

Thus gives that $\frac{x_{n+1}}{x_n} \approx \phi$ (which may be why ϕ shows up in nature.

Suppose young $\to$ old (each young grows up)

old $\to$ old + young (each old has one offspring)