July 11

Eigenvectors

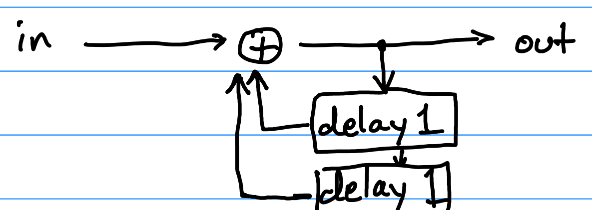
A linear operator is a linear transformation from a vector space V to itself.

ex $\mathcal{C}^\infty(\mathbb{R})$ is the vector space of functions $\mathbb{R} \to \mathbb{R}$ differentiable infinitely many times.
$\frac{d}{dx} : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ is a linear operator.

ex Scaling, rotation, shear, etc. of $\mathbb{R}^2$ are linear operators

We would like to understand linear operators better. For instance, a solution f to the differential equation $f'' - f' - f = 0$ is an element of the kernel of $\left(\frac{d}{dx}\right)^2 - \frac{d}{dx} - 1 : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$.

Another example is a "linear time-invariant system"



For $x_1, x_2, x_3, \ldots$ inputs and $y_1, \ldots$ outputs,

$$y_i = y_{i-1} + y_{i-2} + x_i$$

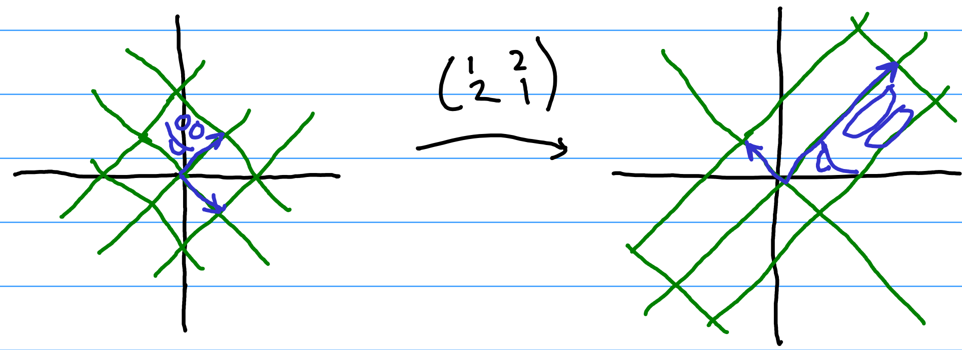
or
$$\begin{bmatrix} y_{i-1} \\ y_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_{i-2} \\ y_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ x_i \end{bmatrix}$$

Suppose the input is constant 0. Then $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is the matrix of a linear operator which transforms a state into the next state for the system.

What we will do: find a basis, if we can, where the action of A is particularly simple — just scaling. Not all operators are amenable, but if they are we will have:

1. a basis P of "eigenvectors"
2. the matrix of A relative to P being diagonal. "eigenvalue" entries.

<u>ex</u> $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = P \begin{pmatrix} 3 & \\ & -1 \end{pmatrix} P^{-1}$, $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$



We will now focus on $V = \mathbb{R}^n$ and an $n \times n$ matrix A, but the definitions easily adapt to linear operators in general.

<u>def</u> An <u>eigenvector</u> $\vec{v} \in \mathbb{R}^n$ for $n \times n$ A is a <u>nonzero</u> vector satisfying $A\vec{v} = \lambda \vec{v}$ for a scalar λ, called the <u>eigenvalue</u> corresponding to $\vec{v}$.

eigen is German for "characteristic"

So: an eigenvector is a nonzero vector A acts on by scaling. We will later see that many matrices have a basis of $\mathbb{R}^n$ of these!

<u>ex</u> $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 3.

How do we find eigenvectors?

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff (A - \lambda I_n)\vec{v} = \vec{0}$$

So: find λ where $(A - \lambda I_n)\vec{v} = \vec{0}$ has a nontrivial solution.

Recall that this solution set is $\mathrm{Nul}(A-\lambda I_n)$.

def The eigenspace for eigenvalue λ of A is $\mathrm{Nul}(A-\lambda I_n)$.
(This is a subspace of $\mathbb{R}^n$.)

Note: λ being an eigenvalue means $\mathrm{Nul}(A-\lambda I_n)$ is not the zero subspace (it contains a nonzero vector: a corresponding eigenvector). Conversely, nonzero vectors of $\mathrm{Nul}(A-\lambda I_n)$ contains nonzero vectors.

The $n\times n$ matrix $A-\lambda I_n$ has a nonzero nullspace if $|A-\lambda I_n|=0$, by the invertible matrix theorem.

The eigenvalues of A are the solutions λ to $|A-\lambda I_n|=0$, the characteristic equation

ex $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$

$$|A-\lambda I_2| = \begin{vmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix} = (1-\lambda)^2-4 = -3-2\lambda+\lambda^2$$
$$= (\lambda-3)(\lambda+1)$$

so $\lambda = 3,-1$.

Eigenvectors:

$\lambda=3$. $\mathrm{Nul}(A-3I_2)$

$$\left[\begin{array}{cc|c}-2&2&0\\2&-2&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]. \qquad \mathrm{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$\lambda=-1$. $\mathrm{Nul}(A+I_2)$

$$\left[\begin{array}{cc|c}2&2&0\\2&2&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right]. \qquad \mathrm{Span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$$

So, for instance, $A\begin{bmatrix}1\\1\end{bmatrix}=3\begin{bmatrix}1\\1\end{bmatrix}$ and $A\begin{bmatrix}1\\-1\end{bmatrix}=-\begin{bmatrix}1\\-1\end{bmatrix}$.

Some eigenspaces are spanned by more than one vector.

ex $A = I_2$. $\quad |A - \lambda I_2| = \begin{vmatrix} 1-\lambda & \\ & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$\lambda = 1$ (multiplicity 2)

$\lambda = 1$, $\mathrm{Nul}(A - I_2) = \mathrm{Nul}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \mathbb{R}^2 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Some matrices have only one dimension of eigenspaces:

ex $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\quad |A - \lambda I_2| = \begin{vmatrix} 1-\lambda & 1 \\ & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$\lambda = 1$ (multiplicity 2)

$\lambda = 1$, $\mathrm{Nul}(A - I_2) = \mathrm{Nul}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Yet others have complex eigenvalues, which we will talk about later (ex: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).

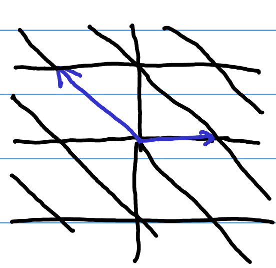
Thm If A is upper triangular, then its eigenvalues are its diagonal entries.

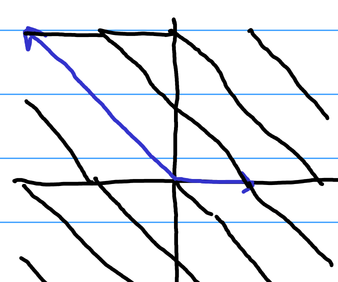
proof $A - \lambda I$ is upper triangular, so $|A - \lambda I|$ is the product of diagonal entries. This is zero exactly when λ is one of those diagonal entries.

ex $A = \begin{pmatrix} 1 & -1 \\ & 2 \end{pmatrix}$ has $\lambda = 1, 2$.

$\lambda = 1$: $\mathrm{Nul}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 2$: $\mathrm{Nul}\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$





Thm If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ are eigenvectors of A, $\lambda_1, \dots, \lambda_k$ corresponding eigenvalues, and $\lambda_1, \dots, \lambda_k$ are distinct, then $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent.

Pf If $\vec{v}_1, \dots, \vec{v}_k$ were instead dependent. There is a smallest p where $\vec{v}_p$ is dependent on $\vec{v}_1, \dots, \vec{v}_{p-1}$, and $\vec{v}_1, \dots, \vec{v}_{p-1}$ are independent. Let $\vec{c} \in \mathbb{R}^{p-1}$ be such that

$$\vec{v}_p = c_1 \vec{v}_1 + \dots + c_{p-1} \vec{v}_{p-1} \qquad (1)$$

Apply A to both sides of (1):

$$A\vec{v}_p = A(c_1 \vec{v}_1 + \dots + c_{p-1} \vec{v}_{p-1})$$
$$A\vec{v}_p = c_1 A \vec{v}_1 + \dots + c_{p-1} A \vec{v}_{p-1}$$
$$\lambda_p \vec{v}_p = c_1 \lambda_1 \vec{v}_1 + \dots + c_{p-1} \lambda_{p-1} \vec{v}_{p-1} \qquad (2)$$

Multiply both sides of (1) by λ_p:

$$\lambda \vec{v}_p = c_1 \lambda_p \vec{v}_1 + \dots + c_{p-1} \lambda_p \vec{v}_{p-1} \qquad (3)$$

Subtract (3) from (2):

$$\vec{0} = c_1(\lambda_1 - \lambda_p)\vec{v}_1 + \dots + c_{p-1}(\lambda_{p-1} - \lambda_p)\vec{v}_{p-1}$$

Since $\vec{v}_1, \dots, \vec{v}_{p-1}$ are independent, each coefficient is zero. The eigenvalues are distinct, so $\lambda_i - \lambda_p \neq 0$, so $c_i = 0$ (with $1 \le i < p$). But, then (1) says $\vec{v}_p = \vec{0}$. This contradicts $\vec{v}_p$ being an eigenvector. ∎

ex $\begin{pmatrix} 1 & 4 & 0 \\ & 2 & 5 \\ & & 3 \end{pmatrix}$ has three distinct eigenvalues. Thus, there is a basis of $\mathbb{R}^3$ consisting only of eigenvectors of this matrix!

$$\lambda = 1: \quad \operatorname{Nul}\begin{pmatrix} 0 & 4 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{pmatrix} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\lambda = 2: \quad \operatorname{Nul}\begin{pmatrix} -1 & 4 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{Span}\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\lambda = 3: \quad \operatorname{Nul}\begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left\{\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}\right\}$$

so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 5 \\ 1 \end{pmatrix}$ is independent spanning set ex $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ are independent because $\begin{pmatrix}1&2\\2&1\end{pmatrix}$ has them as eigenvectors with different eigenvalues.

Eigenvalues generalize nullspace in a way:

An eigenvector with $\lambda=0$ is in the nullspace.

This is because $\lambda=0$ is $A\vec{v}=0\vec{v}=\vec{0}$.

Also, $Nul(A-0I_n)=Nul(A)$.

We extend the Invertible Matrix Theorem:

Thm A is $n\times n$, TFAE

- A is invertible

$\vdots$

- 0 is not an eigenvalue of A

Similarity

Two $n\times n$ matrices A,B are *similar* if $A=PBP^{-1}$ for some invertible P.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[d, "P"'] & \mathbb{R}^n \arrow[d, "P"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

$AP=PB$ is what this diagram says.

In other words, B is the matrix of A relative to some basis P.

Thm If A,B similar then they have the same characteristic polynomial (thus same eigenvalues).

proof
$$\begin{aligned}\det(A-\lambda I_n) &= \det(PBP^{-1}-\lambda I_n)\\ &=\det(PBP^{-1}-\lambda PP^{-1})\\ &=\det(P(B-\lambda I_n)P^{-1})\\ &=\det(P)\det(B-\lambda I_n)\det(P^{-1})\\ &=\det(B-\lambda I_n)\det(PP^{-1})=\det(B-\lambda I_n). \blacksquare\end{aligned}$$

Thm If A, B similar, then the eigenspace for λ of A has the same dimension as that for B.

proof If $A = PBP^{-1}$ and $A\vec{v} = \lambda\vec{v}$, since $BP^{-1} = P^{-1}A$,

$$BP^{-1}\vec{v} = P^{-1}A\vec{v}$$

$$BP^{-1}\vec{v} = P^{-1}(\lambda\vec{v}) = \lambda P^{-1}\vec{v}$$

so $P^{-1}\vec{v}$ is an eigenvector of B. If $\vec{v}_1, \dots, \vec{v}_k$ are independent, so are $P^{-1}\vec{v}_1, \dots, P^{-1}\vec{v}_k$, so
$\dim \operatorname{Nul}(A - \lambda I_n) \leq \dim \operatorname{Nul}(B - \lambda I_n)$. Swapping the roles of A, B finishes the equality.

ex $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ are <u>not</u> similar despite

having the same characteristic equation.
2-dim for $\lambda = 1$ vs 1-dim.

ex The only matrix similar to $\begin{pmatrix} c & \\ & c \end{pmatrix}$ is $P\begin{pmatrix} c & \\ & c \end{pmatrix}P^{-1} = \begin{pmatrix} c & \\ & c \end{pmatrix}$.
Makes sense since 2d e.space for c is $\mathbb{R}^2$ itself, so any similar matrix to it must do the same.

ex $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\lambda = 3, 1$ $\qquad\qquad$ $\lambda = 1, 1$

so row ops do <u>not</u> preserve eigenvalues or similarity

Next time Find a diagonal matrix A is similar to (if one exists)