July 8

Last time:

- defined "finite dimensional vector space" (has a finite basis)
- found the "isomorphism classes" (one per dimension, represented by $\mathbb{R}^n$)
- basis is maximal independent set or minimal spanning set
- all bases of V have the same # of vectors.

Rank

We will take a look at matrices again, using subspace concepts.

Question: # pivots in A = # pivots in A^T?
(if true, somehow rref(A) and rref(A^T) are closely related!)

Partial answer: if A is square $(n\times n)$, by the invertible matrix theorem, if A doesn't have n pivots, neither does A^T.

We will have #pivots as a measure of how non-invertible a matrix is, so to speak.

A row of an $m\times n$ matrix A can be thought of as vectors in $\mathbb{R}^n$. That is, a row of A is a column of $n\times m$ A^T.

def $\text{Row } A = \text{Col } A^T$. (the span of the rows of A treated as column vectors)

ex $$\text{Row}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}\right\}$$

To find a basis for Row A, we could perform rref(A^T), find the pivot columns, and take those rows of A. However, we can do better (given we want to relate Col A to Row A by rref):

Thm If $A \sim B$, Row A = Row B. In fact, the nonzero rows of a ref(A) are a basis of Row A

Proof Row operations applied to A replace rows with linear combinations of rows of A, so Row (A) $\subset$ Row B. Since $B \sim A$, too, Row (B) $\subseteq$ Row A. Thus, Row A = Row B.
In a ref of A, pivots occur in different columns, so no nonzero row is a linear combination of the ones below it.
Thus, the nonzero rows of a ref(A) are independent and span Row (A). This is a basis. $\blacksquare$

Thus, rref (A) gives:
1) dim Nul (A), as # free columns (basis is from param. vec. form)
2) dim Col (A), as # pivot columns (basis is those cols of A)
3) dim Row (A), as # pivot rows (basis is those rows of rref(A))

ex $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

so $\quad \text{Nul } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$\text{Col } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

$\text{Row } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$

So: dim Row A = dim Col A

Since Row A = Col A^T, we have answered our question:

$$\dim \operatorname{Col} A^T = \dim \operatorname{Col} A$$
$$(\#\text{ pivots in } A^T = \#\text{ pivots in } A)$$

We define the <u>rank</u> of A to be the # of pivots of A.
More intrinsically, rank A = dim Col A
← i.e., doesn't depend on some rref

Sometimes, dim Nul A is called <u>nullity</u>. Hence, the following theorem's name:

<u>Thm (Rank-nullity)</u> For $m \times n$ matrix A,

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

<u>Proof</u> Every column is either a pivot column or a free column.
(Remember: Nul A is solutions to $A\vec{x} = \vec{0}$)

<u>ex</u> A is 3×5. The following dimensions are possible:

rank A	dim Nul A
0	5
1	4
2	3

dim Nul $A \geq 3$, so new way to see columns must be dependent!

ex A is 5×3

$\text{rank } A$	$\dim \text{Nul } A$
3	0
2	1
0	3

so, $\text{rank } A \leq 3$, so $\text{Col } A$ cannot be all of $\mathbb{R}^5$.

ex A $m\times n$, $\text{rank } A = \text{rank } A^T$, so

$$\text{rank } A + \dim \text{Nul } A = n$$
$$\text{rank } A + \dim \text{Nul } A^T = m$$

so $\dim \text{Nul } A^T = m - n + \dim \text{Nul } A$

if $m=n$, $\dim \text{Nul } A = \dim \text{Nul } A^T$.

We may extend the invertible matrix theorem:

Thm A is $n\times n$, the following are equivalent:

- A is invertible

  ⋮
- $\text{rank } A = n$ ("full rank")
- $\dim \text{Col } A = n$
- $\text{Col } A = \mathbb{R}^n$
- $\dim \text{Nul } A = 0$
- $\text{Nul } A = \{\vec{0}\}$

(Row A statements are Col A^T statements, so omitted)

Note: rank determination is treacherous. Any mistake will result in a likely misidentification of rank.

for example, $\begin{vmatrix} 1 & 2 \\ 2 & x \end{vmatrix} = x-4$, which is nonzero for almost all x.

In fact, a random matrix almost surely has full rank.

Change of basis

Coordinates are with respect to a particular basis. What if we want to change between bases?

Let $\mathcal{B} : \mathbb{R}^n \to V$ and $\mathcal{C} : \mathbb{R}^n \to V$ be two bases.

$$\begin{tikzcd} \mathbb{R}^n \arrow[dr, "\mathcal{B}"] & & \mathbb{R}^n \arrow[dl, "\mathcal{C}"'] \\ & V & \end{tikzcd}$$

There is a linear transformation from $\mathcal{B}$-coords to $\mathcal{C}$-coords given by $T(\vec{x}) = \mathcal{C}^{-1}(\mathcal{B}(\vec{x}))$. This is $\mathbb{R}^n \to \mathbb{R}^n$, so it can be given by a matrix. Let $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ be $n \times n$ matrix $[T]$.

ex $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ two bases of $\mathbb{R}^2$

$$[T] = \left[\, T(\vec{e}_1) \quad T(\vec{e}_2) \,\right].$$

$$T(\vec{e}_1) = \mathcal{C}^{-1}\mathcal{B}\vec{e}_1 = \mathcal{C}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

$$T(\vec{e}_2) = \mathcal{C}^{-1}\mathcal{B}\vec{e}_2 = \mathcal{C}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{so } \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 \\ -1/2 & 0 \end{bmatrix}$$

Thus, coordinate $\begin{bmatrix}1\\1\end{bmatrix}$ w.r.t. $\mathcal{B}$ is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\-1/2\end{bmatrix} \text{ w.r.t. } \mathcal{C}.$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}"] \arrow[dr, "\mathcal{B}"'] & & \mathbb{R}^n \arrow[dl, "\mathcal{C}"] \\
& V &
\end{tikzcd}$$

$$\mathcal{B} = \mathcal{C}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$

or $$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \mathcal{C}^{-1}\mathcal{B}$$

(can take any path)

For $\mathcal{C} = I_n$ for $V = \mathbb{R}^n$,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \mathcal{C}^{-1}\mathcal{B} = \mathcal{B} \quad \text{(the vector for coord.)}$$

For $\mathcal{B} = I_n$,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \mathcal{C}^{-1} \quad \text{(the coord for } \vec{x} \text{ rel } \mathcal{C}\text{)}$$

ex $\mathcal{B} = (1 \quad 1+x) \qquad \mathcal{C} = (1+x \quad 1-x)$

$$\left.\begin{aligned}
\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\vec{e}_1 &= \mathcal{C}^{-1}\mathcal{B}\vec{e}_1 = \mathcal{C}^{-1}(1) = \begin{pmatrix}1/2\\1/2\end{pmatrix} \\
\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\vec{e}_2 &= \mathcal{C}^{-1}\mathcal{B}\vec{e}_2 = \mathcal{C}^{-1}(1+x) = \begin{pmatrix}1\\0\end{pmatrix}
\end{aligned}\right\} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix}1/2 & 1\\1/2 & 0\end{pmatrix}.$$