

July 8

Last time:

- defined "finite dimensional vector space" (has a finite basis)
- found the "isomorphism classes" (one per dimension, represented by \mathbb{R}^n)
- basis is maximal independent set or minimal spanning set
- all bases of V have the same # of vectors.

Rank

We will take a look at matrices again, using subspace concepts.

Question: # pivots in $A =$ # pivots in A^T ?
(if true, somehow $\text{rref}(A)$ and $\text{rref}(A^T)$ are closely related!)

Partial answer: if A is $(n \times n)$ square, by the invertible matrix theorem, if A doesn't have n pivots, neither does A^T .

We will have # pivots as a measure of how non-invertible a matrix is, so to speak.

A row of an $m \times n$ matrix A can be thought of as vectors in \mathbb{R}^n . That is, a row of A is a column of $n \times m$ A^T .

def $\text{Row } A = \text{Col } A^T$. (the span of the rows of A treated as column vectors)

ex $\text{Row} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

To find a basis for Row A , we could perform $\text{rref}(A^T)$, find the pivot columns, and take those rows of A . However, we can do better (given we want to relate Col A to Row A by rref):

Thus If $A \sim B$, $\text{Row } A = \text{Row } B$. In fact, the nonzero rows of a $\text{rref}(A)$ are a basis of Row A .

Proof Row operations applied to A replace rows with linear combinations of rows of A , so $\text{Row}(A) \subseteq \text{Row } B$. Since $B \sim A$, too, $\text{Row}(B) \subseteq \text{Row } A$. Thus, $\text{Row } A = \text{Row } B$.

In a rref of A , pivots occur in different columns, so no nonzero row is a linear combination of the ones below it.

Thus, the nonzero rows of a $\text{rref}(A)$ are independent and span Row (A) . This is a basis. \square

Thus, $\text{rref}(A)$ gives:

1) $\dim \text{Nul}(A)$, as # free columns (basis is from param. vec. form)

2) $\dim \text{Col}(A)$, as # pivot columns (basis is those cols of A)

3) $\dim \text{Row}(A)$, as # pivot rows (basis is those rows of $\text{rref}(A)$)

ex $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

so $\text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$\text{Col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

$\text{Row } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

So: $\dim \text{Row } A = \dim \text{Col } A$

Since $\text{Row } A = \text{Col } A^T$, we have answered our question:

$$\begin{aligned} \dim \text{Col } A^T &= \dim \text{Col } A \\ (\# \text{ pivots in } A^T &= \# \text{ pivots in } A) \end{aligned}$$

We define the rank of A to be the # of pivots of A .
More intrinsically, $\text{rank } A = \dim \text{Col } A$
↑ i.e., doesn't depend on some ref

Sometimes, $\dim \text{Nul } A$ is called nullity. Hence, the following theorem's name:

Thm (Rank-nullity) For $m \times n$ matrix A ,
 $\text{rank } A + \dim \text{Nul } A = n$.

Proof Every column is either a pivot column or a free column.
(Remember: $\text{Nul } A$ is solutions to $A\vec{x} = \vec{0}$)

ex A is 3×5 . The following dimensions are possible:

<u>rank</u> A	<u>dim</u> $\text{Nul } A$
0	5
1	4
2	3

$\dim \text{Nul } A \geq 3$, so new way to see columns must be dependent!

ex A is 5×3

rank A	dim Nul A
3	0
2	1
0	3

so, rank $A \leq 3$, so Col A cannot be all of \mathbb{R}^5 .

ex A $m \times n$, rank $A = \text{rank } A^T$, so

$$\text{rank } A + \text{dim Nul } A = n$$

$$\text{rank } A + \text{dim Nul } A^T = m$$

$$\text{so } \text{dim Nul } A^T = m - n + \text{dim Nul } A$$

$$\text{if } m = n, \text{dim Nul } A = \text{dim Nul } A^T.$$

We may extend the invertible matrix theorem:

Thm A is $n \times n$, the following are equivalent:

• A is invertible

⋮

• rank $A = n$ ("full rank")

• $\text{dim Col } A = n$

• $\text{Col } A = \mathbb{R}^n$

• $\text{dim Nul } A = 0$

• $\text{Nul } A = \{\vec{0}\}$

(Row A statements are $\text{Col } A^T$ statements, so omitted)

Note: rank determination is treacherous. Any mistake will result in a likely misidentification of rank.

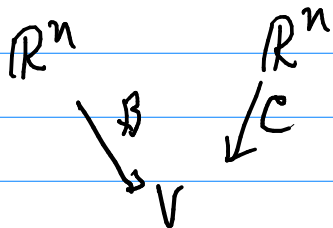
for example, $\begin{vmatrix} 1 & 2 \\ 2 & x \end{vmatrix} = x - 4$, which is nonzero for almost all x .

In fact, a random matrix almost surely has full rank.

Change of basis

Coordinates are with respect to a particular basis. What if we want to change between bases?

Let $B: \mathbb{R}^n \rightarrow V$ and $C: \mathbb{R}^n \rightarrow V$ be two bases.



There is a linear transformation from B -coords to C -coords given by $T(\vec{x}) = C^{-1}(B(\vec{x}))$. This is $\mathbb{R}^n \rightarrow \mathbb{R}^n$, so it can be given by a matrix. Let $P_{C \leftarrow B}$ be $n \times n$ matrix $[T]$.

ex $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ two bases of \mathbb{R}^2

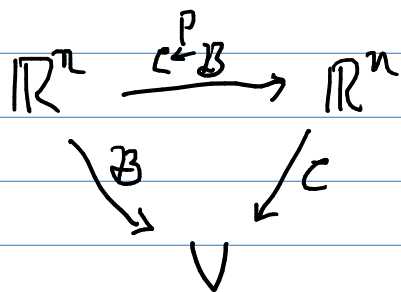
$$[T] = [T(\vec{e}_1) \quad T(\vec{e}_2)]$$

$$T(\vec{e}_1) = C^{-1}B\vec{e}_1 = C^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$$T(\vec{e}_2) = C^{-1}B\vec{e}_2 = C^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{so } P_{C \leftarrow B} = \begin{bmatrix} 1 & 1 \\ -1/2 & 0 \end{bmatrix}$$

Thus, coordinate $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w.r.t. \mathcal{B} is

$$P_{\mathcal{C} \leftarrow \mathcal{B}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix} \text{ w.r.t. } \mathcal{C}.$$



$$\mathcal{B} = C P_{\mathcal{C} \leftarrow \mathcal{B}}$$

$$\text{or } P_{\mathcal{C} \leftarrow \mathcal{B}} = C^{-1} \mathcal{B}$$

(can take any path)

For $\mathcal{C} = I_n$ for $V = \mathbb{R}^n$,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \mathcal{C}^{-1} \mathcal{B} = \mathcal{B} \quad (\text{the vector for coord.})$$

For $\mathcal{B} = I_n$,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \mathcal{C}^{-1} \quad (\text{the coord for } \vec{x} \text{ rel } \mathcal{C})$$

ex $\mathcal{B} = \begin{pmatrix} 1 & 1+x \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 1+x & 1-x \end{pmatrix}$

$$\left. \begin{aligned} P_{\mathcal{C} \leftarrow \mathcal{B}} \vec{e}_1 &= \mathcal{C}^{-1} \mathcal{B} \vec{e}_1 = \mathcal{C}^{-1}(1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \\ P_{\mathcal{C} \leftarrow \mathcal{B}} \vec{e}_2 &= \mathcal{C}^{-1} \mathcal{B} \vec{e}_2 = \mathcal{C}^{-1}(1+x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \right\} P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}.$$