July 7

Coordinates

What a basis does for a vector space V is to let us specify vectors, uniquely, using $\mathbb{R}^n$. This is like laying down axes in the space to find our way in it.

Let $\vec{b}_1, \cdots, \vec{b}_n \in V$ be vectors with
$$\mathcal{B} = (\vec{b}_1 \; \cdots \; \vec{b}_n)$$
a basis (written as a hypervector). Recall that
$$\mathcal{B}\vec{c} = c_1\vec{b}_1 + \cdots + c_n\vec{b}$$
is how we write linear combinations. The two properties of a basis let us say the following:

1. Every $\vec{x} \in V$ has at least one $\vec{c} \in \mathbb{R}^n$ with $\vec{x} = \mathcal{B}\vec{c}$. This is because $\vec{b}_1, \cdots, \vec{b}_n$ span V.
2. Every $\vec{x} \in V$ has at most one such $\vec{c}$. If $\vec{c}, \vec{d} \in \mathbb{R}^n$ with $\mathcal{B}\vec{c} = \vec{x} = \mathcal{B}\vec{d}$, then $\mathcal{B}\vec{c} - \mathcal{B}\vec{d} = \vec{0}$, so $\mathcal{B}(\vec{c} - \vec{d}) = \vec{0}$. Because $\mathcal{B}$ is independent, $\vec{c} - \vec{d} = \vec{0}$, so $\vec{c} = \vec{d}$.

That is, every vector $\vec{x}$ in V can be represented in exactly one way as $\mathcal{B}\vec{c}$, with $\vec{c}$ the <u>coordinate vector of $\vec{x}$ relative to $\mathcal{B}$</u>. This is called "unique representation."

$\mathcal{B} : \mathbb{R}^n \to V$ is a linear transformation which is onto (1) and one-to-one (2), so it is invertible as a function. Given $\vec{x} \in V$, $\mathcal{B}^{-1}(\vec{x}) = \vec{c}$ is the coordinate such that $\mathcal{B}\vec{c} = \vec{x}$. By inverse, we mean $\mathcal{B}(\mathcal{B}^{-1}(\vec{x})) = \vec{x}$ and $\mathcal{B}^{-1}(\mathcal{B}(\vec{c})) = \vec{c}$.

The book uses the older notation $[\vec{x}]_{\mathcal{B}}$ for $\mathcal{B}^{-1}\vec{x}$, so $[c_1\vec{b}_1 + \cdots + c_n\vec{b}_n]_{\mathcal{B}} = \vec{c}$. The downside is that it obscures that it is a linear transformation, and that function composition is more difficult to write nicely.

ex $\mathcal{B} = (1+x \quad 1-x)$ is a basis for $\mathbb{P}_1$.

$$\mathcal{B}^{-1}(x) = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \text{ since } \mathcal{B}\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x) = x.$$

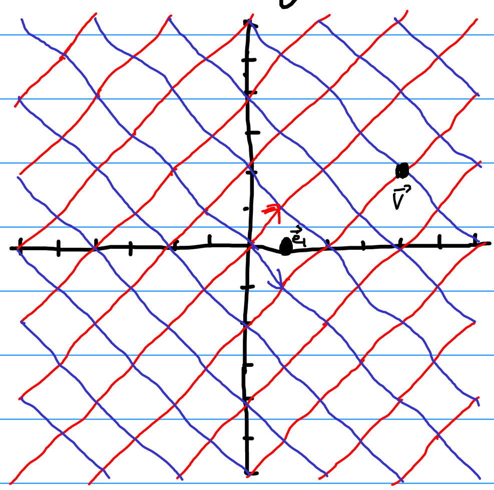
ex $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a basis for $\mathbb{R}^2$

$\mathcal{B}^{-1}(\vec{e}_1) = \vec{c}$ is calculated by solving $\vec{e}_1 = \mathcal{B}\vec{c}$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right)$$

so $\vec{c} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. In fact, $\mathcal{B}\vec{c} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What we did was locate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ according to the axes given by $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$



From this diagram, we can also see $\vec{v} = \mathcal{B}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, or $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \mathcal{B}^{-1}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

The standard axes correspond to the standard basis.

For $\mathbb{R}^n$, a basis is equivalent to the columns of an invertible matrix. The textbook has $P_{\mathcal{B}}$ being this $n \times n$ matrix, but $\mathcal{B} = (\vec{b}_1 \ \cdots \ \vec{b}_n)$ works just as well.

Sometimes it is convenient to organize bases in diagrams involving arrows:

$$\mathbb{R}^n \xrightarrow{\mathcal{B}} V$$

"coordinate space"

This is merely saying $\mathcal{B}: \mathbb{R}^n \to V$, but it will prove useful when other bases or transformations are involved, for instance

$$\begin{tikzcd} \mathbb{R}^n \arrow[dr, "\mathcal{B}"'] & & \mathbb{R}^n \arrow[dl, "\mathcal{C}"] \\ & V & \end{tikzcd}$$

Also, if $V = \mathbb{R}^n$, too, then $\mathbb{R}^n \xrightarrow{\mathcal{B}} \mathbb{R}^n$ has the first $\mathbb{R}^n$ the coordinates for the second $\mathbb{R}^n$. This can get confusing.

We said $\mathcal{B}: \mathbb{R}^n \to V$ is one-to-one and onto. Whenever a linear transformation is both, it is called an isomorphism for same-form.

ex $\mathcal{B} = (1 \quad 1+x \quad 1+x+x^2)$ is a basis for $\mathbb{P}_2$. Let us add x to x^2 in $\mathbb{P}_2$ using the isomorphism with $\mathbb{R}^3$.

$$\mathcal{B}^{-1}(x) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B}^{-1}(x^2) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathcal{B}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 + 1+x+x^2 = x+x^2.$$

Adding in $\mathbb{R}^3$ corresponded to adding in $\mathbb{P}_2$!

ex. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be given by $p \mapsto p'$.
Let $\mathcal{B} = (1 \;\; x \;\; x^2)$ be standard basis.

$[T]_{\mathcal{B}}$ is the <u>matrix of T relative to $\mathcal{B}$</u>
satisfying $T(p) = \mathcal{B}\,[T]_{\mathcal{B}}\,\mathcal{B}^{-1}(p)$, represented as

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "{[T]_{\mathcal{B}}}"] \arrow[d, "\mathcal{B}"] & \mathbb{R}^3 \arrow[d, "\mathcal{B}"] \\
\mathbb{P}_2 \arrow[r, "T"] \arrow[r, bend left=80, "{\mathcal{B}[T]_{\mathcal{B}}\mathcal{B}^{-1}}"] & \mathbb{P}_2
\end{tikzcd}$$

I am being cavalier in my treating a matrix as a transformation. This is OK as long as you know that we really mean the transf. def. by mult. by the matrix.

We can calculate this matrix by observing that also $\mathcal{B}^{-1} T \mathcal{B} = [T]_{\mathcal{B}}$ is a transformation $\mathbb{R}^3 \to \mathbb{R}^3$, so we can find its standard matrix.

Column 1: $\mathcal{B}^{-1} T \mathcal{B} \vec{e}_1 = \mathcal{B}^{-1} T(1) = \mathcal{B}^{-1} 0 = \vec{0}$
Column 2: $\mathcal{B}^{-1} T \mathcal{B} \vec{e}_2 = \mathcal{B}^{-1} T(x) = \mathcal{B}^{-1} 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
Column 3: $\mathcal{B}^{-1} T \mathcal{B} \vec{e}_3 = \mathcal{B}^{-1} T(x^2) = \mathcal{B}^{-1}(2x) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

So,
$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}\,[T]_{\mathcal{B}} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So $\ker T = \text{Span}\left\{ \mathcal{B} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Span}\{1\}$
(all constant polys)

and $\text{im}\, T = \text{Span}\left\{ \mathcal{B} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathcal{B} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\} = \text{Span}\{1, 2x\}$
(all deg ≤ 1 polys).

Dimension

We will show that every basis of a vector space has the same number of vectors. Since this is constant, we give it a name, dimension.

Thm If $\vec{v}_1, \dots, \vec{v}_n \in V$ span V, and $\vec{w}_1, \dots, \vec{w}_m \in V$ with $m > n$, then $\vec{w}_1, \dots, \vec{w}_m$ are dependent.

Proof Each $\vec{w}_i$ is in the span, so there is some $\vec{x}_i \in \mathbb{R}^n$ with $\vec{w}_i = (\vec{v}_1 \cdots \vec{v}_n)\vec{x}_i$.

The matrix $(\vec{x}_1 \cdots \vec{x}_m)$ has more columns than rows, so the vectors are dependent, with $(\vec{x}_1 \cdots \vec{x}_m)\vec{c} = \vec{0}$.

Now, $(\vec{w}_1 \cdots \vec{w}_m) = (\vec{v}_1 \cdots \vec{v}_n)(\vec{x}_1 \cdots \vec{x}_m)$

so
$$(\vec{w}_1 \cdots \vec{w}_m)\vec{c} = (\vec{v}_1 \cdots \vec{v}_n)(\vec{x}_1 \cdots \vec{x}_m)\vec{c}$$
$$= (\vec{v}_1 \cdots \vec{v}_n)\vec{0} = \vec{0}.$$

Thus, $(\vec{w}_1 \cdots \vec{w}_m)\vec{c} = \vec{0}$ is a dependence. ▨

This means that if $\mathcal{B}, \mathcal{C}$ are two bases of a vector space V, then if one has more vectors than the other, then since they both span V, one must be dependent — this contradicts their both being bases!

Consequence If $V \to W$ isomorphism, $\mathbb{R}^n \xrightarrow{\mathcal{B}} V$ basis, $\mathbb{R}^m \xrightarrow{\mathcal{C}} W$ basis, $n = m$.
"Invariance of dimension."

The spanning set theorem let us produce bases from spanning sets, so this says no matter how we remove dependent vectors, we will always end up with the same number of vectors in a basis!

A vector space is finite dimensional if it has a basis (of finitely many elements). Otherwise, it is infinite dimensional. $\mathbb{R}^n$ and $\mathbb{P}_n$ are finite-dimensional. $\mathbb{P}$ is not. We write dim V for the dimension of V.

As a dual to basis-from-span, we have the following, too:

thm (Basis extension) If V is finite-dimensional, W subspace, and if $\vec{v}_1, \dots, \vec{v}_k \in W$ are independent, then there is a basis of W including these vectors.

proof Case I: they span W. Then they are already a basis.

Case II: they don't. Then there is a $\vec{v}_{k+1} \in W$ outside $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Consider $(\vec{v}_1 \cdots \vec{v}_{k+1})\vec{c} = \vec{0}$.

If $\vec{c}_{k+1} = \vec{0}$, by independence of $\vec{v}_1, \dots, \vec{v}_k$, $c_1 = \cdots = c_k = 0$, too.

Else, $\vec{v}_{k+1}$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_k$, but $\vec{v}_{k+1}$ is not in the span!

Thus, $\vec{c} = \vec{0}$, so $\vec{v}_1, \dots, \vec{v}_{k+1}$ are independent.

Re-apply the theorem.

If we could keep finding new vectors, eventually we would have more than dim V, which would imply they are dependent. Thus, we end up with a basis.

A consequence of this is every such subspace is a span of finitely many vectors!

ex Extend $\begin{pmatrix}1\\1\\1\end{pmatrix}$ to a basis of $\mathbb{R}^3$.

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \notin \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \notin \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$

so $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ is an extended basis.

What this theorem implies is that whenever W is a subspace of a finite-dimensional V,

$$\dim W \leq \dim V$$

One more important theorem:

Basis theorem V is finite dimensional vector space, and $n = \dim V$.
(i) A collection of n lin. indep. vectors in V is a basis.
(ii) A collection of n vectors spanning V is a basis.

proof (i) Extend the vectors to a basis of V. No vectors were added since all bases have n vectors.
(ii) Remove vectors until basis. No vectors were removed since all bases have n vectors.

ex Compute $\dim \operatorname{Nul}\begin{pmatrix}1 & 2 & 3 & 5 & 0\\0 & 1 & 4 & 6 & 0\\0 & 0 & 0 & 1 & 7\end{pmatrix}$.
There are 2 free columns, so any basis has 2 vectors.
$\dim \operatorname{Nul} A = 2$.

No need to actually find the basis!

In fact, we have the following important result:

Rank-nullity theorem A is $m \times n$:

$$\underset{\text{"rank"} = \#\text{pivots}}{\dim \operatorname{Col} A} + \underset{\text{"nullity"}}{\dim \operatorname{Nul} A} = n.$$

ex A is 4×3 and its columns span a 2-dim. subspace. Is $\vec{x} \mapsto A\vec{x}$ one-to-one?

$$3 + \dim \operatorname{Nul} A = 3$$

So $\dim \operatorname{Nul} A = 0$. That is, $\operatorname{Nul} A = \{\vec{0}\}$.
Thus, $\vec{x} \mapsto A\vec{x}$ is one-to-one.

Further examples V, W finite-dimensional vector spaces.

- $T: V \to W$, $\dim V \geq \dim \operatorname{im} T$.

  Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis of V
  then $T(\vec{v}_1), \dots, T(\vec{v}_n)$ span $\operatorname{im} T$

- $T: V \to W$, T onto. Then $\dim V \geq \dim W$
  $(\operatorname{im} T = W)$
- $T: V \to W$, T one-to-one. Then $\dim V \leq \dim W$.
  If $\vec{v}_1, \dots, \vec{v}_n$ basis for V, suppose $\vec{c} \in \mathbb{R}^n$ such that
  $c_1 T\vec{v}_1 + \dots + c_n T\vec{v}_n = \vec{0}$
  $\Rightarrow T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n) = \vec{0}$. One-to-one $\Rightarrow c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$.
  Basis $\Rightarrow \vec{c} = \vec{0}$. Thus W has at least n indep. vectors.
- $T: V \to W$ isomorphism. Then $\dim V = \dim W$.