Bases

Our goal is to describe vector spaces as the span of some collection of vectors — a minimal such set will be called a "basis". This will allow us to study vector spaces as if they were $\mathbb{R}^n$, which will grant us the ability to calculate with coordinates.

A hypervector or indexed set is an ordered collection of vectors $(\vec{a}_1 \; \cdots \; \vec{a}_n)$ with $\vec{a}_1, \ldots, \vec{a}_n \in V$, where V is a vector space.

A linear combination of $\vec{a}_1, \ldots, \vec{a}_n$ with a coordinate vector or vector of weights $\vec{c} \in \mathbb{R}^n$ is $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n$, also written

$$(\vec{a}_1 \; \cdots \; \vec{a}_n)\,\vec{c}$$

An $m \times n$ matrix is a special case of hypervector, where each of the n vectors is from $\mathbb{R}^m$.

An indexed set of vectors $\vec{a}_1, \ldots, \vec{a}_n$ is linearly independent if $(\vec{a}_1 \; \cdots \; \vec{a}_n)\,\vec{c} = \vec{0}$ has only the trivial solution $\vec{c} = \vec{0}$. Otherwise, they are linearly dependent. A nonzero solution $\vec{c}$ written as

$$c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{0}$$

is called a dependence relation among $\vec{a}_1, \ldots, \vec{a}_n$.

All of this is simply a generalization from $\mathbb{R}^m$ to a general vector space V.

ex $1, x, x^2 \in \mathbb{P}$ are independent.

If $c_1 1 + c_2 x + c_3 x^2 = 0$ for all x, then every x is a root of the left-hand side. A quadratic has two roots, so $c_3 = 0$. Linear polynomials have one, so $c_2 = 0$. Finally, $c_1 = 0$. Thus, $(1 \quad x \quad x^2)\,\vec{c} = 0$ has only the trivial solution.

ex. $\cos(2x)$, 1, and $\cos^2 x$ are dependent.

Since $\cos^2 x = \dfrac{1 + \cos 2x}{2}$, we have a dependence

$$\cos(2x) + 1 - 2\cos^2 x = 0$$

or, $$\left(\cos(2x) \quad 1 \quad \cos^2 x\right)\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0.$$

A useful fact about dependence is that one of the vectors is then a linear combination of the rest. Say $(\vec{a}_1 \cdots \vec{a}_n)\vec{c} = \vec{0}$ with $\vec{c} \neq 0$. Let j be where c_j is the last nonzero weight. Thus,

$$c_1 \vec{a}_1 + \cdots + c_j \vec{a}_j = \vec{0}.$$

Now, since $c_j \neq 0$,

$$\vec{a}_j = -\frac{c_1}{c_j}\vec{a}_1 - \frac{c_2}{c_j}\vec{a}_2 - \cdots - \frac{c_{j-1}}{c_j}\vec{a}_{j-1}.$$

In the previous example, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

another ex $\sin x$ and $\cos x$ are independent.

If $c_1 \sin x + c_2 \cos x = 0$,

$x = 0$ means

$$0 + c_2 = 0$$

$x = \pi/2$ means

$$c_1 + 0 = 0$$

so $c_1 = c_2 = 0$. $\vec{c}$ is the trivial solution.

def Let V be a vector space. A basis is an indexed set of vectors $\vec{v}_1, \dots, \vec{v}_n \in V$ which

1) are linearly independent
2) span V (i.e., $V = \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$

"a basis is a linearly independent spanning set."

(This definition applies just as well to subspaces, which we will focus on, because they are vector spaces in their own right.)

ex If $A = (\vec{a}_1 \ \cdots \ \vec{a}_n)$ is an invertible matrix (n pivots) then $\vec{a}_1 \cdots \vec{a}_n$ are a basis for $\mathbb{R}^n$. Vice versa, too.

ex $I_n = (\vec{e}_1 \ \cdots \ \vec{e}_n)$ is the standard basis for $\mathbb{R}^n$.

ex A collection of $> n$ vectors of $\mathbb{R}^n$ must be dependent, and $< n$ must not span. Bases then have n vectors.

For a similar reason as in a previous example, $(1 \;\; x \;\; x^2 \;\; \cdots \;\; x^n)$ is a basis for $\mathbb{P}_n$ (the degree-at-most-n polynomial). This is the standard basis for $\mathbb{P}_n$.

The spanning set theorem

An indexed set of vectors which spans V can be simplified until it becomes an independent set, while retaining spanning-ness. This is a way to make a basis.

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ are dependent. We may swap pairs of vectors until weight $c_n \neq 0$, and we may multiply the dependence by $\frac{1}{c_n}$, so we may assume $c_n = 1$. Thus, $\vec{v}_n = -c_1\vec{v}_1 - \cdots - c_{n-1}\vec{v}_{n-1}$.

Given a linear combination $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$, we may substitute the above for $\vec{v}_n$ to get

$$d_1\vec{v}_1 + \cdots + d_{n-1}\vec{v}_{n-1} + d_n(-c_1\vec{v}_1 - \cdots - c_{n-1}\vec{v}_{n-1})$$
$$= (d_1 - c_1 d_n)\vec{v}_1 + \cdots + (d_{n-1} - c_{n-1}d_n)\vec{v}_{n-1}$$

so it is in fact a linear combination of the first $n-1$.

So, either $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent (and thus a basis) or $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$. We can't keep removing vectors, since n is finite, so eventually we will obtain a basis.

Thm (Spanning set theorem) If $\vec{v}_1, \ldots, \vec{v}_n$ span V, then either they are a basis, or one can be removed while still spanning V.

That is,

for every spanning set, there is a subset which is a basis.

ex • A basis for $\mathrm{Col}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

Since $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ is a nontrivial homogeneous solution, the third column is a linear combination of the first two, so $\mathrm{Col}\, A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

• A basis for $\mathrm{Col}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

The second column is a lin. comb. of the first, so again, $\mathrm{Col}\, A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

In general, columns are independent if $A\vec{x} = \vec{0}$ has only trivial solution, else get a dependence. The way we have been solving $[A \mid \vec{0}]$ in parametric vector form gives one vector per free column, with 1 in the row corresponding to the free column and 0's below. Thus, the free columns of A are linear combinations of the preceding columns!

Col A is the span of the pivot columns of A. These columns are a basis of Col A

Warning These are columns of $\underline{A}$ not rref(A).

ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

so columns 1, 2 pivot columns, so

$$\text{Col } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$

$\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$ is a basis of Col A.

As for Nul(A), the vectors obtained from parametric vector form have the property that the rows corresponding to free columns form an identity matrix, so they are independent. Thus they are already a basis!

ex $\text{Nul}\begin{pmatrix} 1 & 2 & 0 & 3 \\ & & 1 & 4 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Notice: the number of vectors in a basis for Col A = # pivots
the number of vectors in a basis for Nul A = # free cols

pivots + # free cols = # columns.

This is something to remember.

Dual view

We took spanning sets and reduced them to a basis. This is minimal in the sense that removing any more vectors gives an independent set which no longer spans V.

We can also take an independent set and extend it to a basis. For $\vec{v}_1, \ldots, \vec{v}_n \in V$ which are independent and do not span V, let $\vec{v}_{n+1} \in V$ be a vector outside their span. It is thus independent.

Unlike for reduction, it is not obvious that extending will ever terminate. For instance, for $\mathbb{P}$, $1, x, x^2, x^3, \ldots$ are independent, so in fact it might not. This takes a combination of Zorn's lemma (outside the scope of this course) and invariance of dimension to be able to conclude the process will terminate for $\mathbb{R}^n$, at least. Basis extension is useful, albeit not in the book.

ex $\operatorname{Nul}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$, which we extend to

$\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ as a basis for $\mathbb{R}^2$.

$$A\left(x_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = x_1 A\begin{pmatrix} -2 \\ 1 \end{pmatrix} + x_2 A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \vec{0} + x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so every vector in $\mathbb{R}^2$ is the sum of a vector of $\operatorname{Nul} A$ and a vector whose image is not $\vec{0}$.

Purpose of a basis.

1) span V. That is, every vector of V can be expressed as a linear combination in <u>at least one</u> way

2) independent. Every vector can be expressed in <u>at most one</u> way.

So, together, exactly one way.

Thus, $\vec{v}_1, \ldots, \vec{v}_n$ are a basis if and only if

$\mathcal{B} : \mathbb{R}^n \to V$ def. by $\vec{c} \mapsto (\vec{v}_1 \cdots \vec{v}_n)\vec{c}$

is one-to-one (2) and onto (1).

(Note: bases are not unique. Basis change is about finding a matrix P so that $\mathcal{B} = \mathcal{C}P$)

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "P"] \arrow[dr, "\mathcal{B}"'] & & \mathbb{R}^n \arrow[dl, "\mathcal{C}"] \\ & V & \end{tikzcd}$$

(expresses the above equality)