July 5

Vector space review

A real vector space is a set V of "vectors", a way to add them, a way to scalar multiply by $\mathbb{R}$, and eight coherence properties which make sure the operations are $\mathbb{R}^n$-like.

Go-to examples: $\mathbb{R}^n$, polynomials, (continuous) $\mathbb{R}$-valued functions.

Most vector spaces we care about are "subspaces" of a given vector space:

def A subset W of a vector space V is a <u>subspace</u> if

(a) W has <u>closure under addition</u> (for $\vec{u},\vec{v} \in W$, $\vec{u}+\vec{v} \in W$)
(b) W has <u>closure under scalar multiplication</u> (for $c \in \mathbb{R}$ and $\vec{v} \in W$, $c\vec{v} \in W$)

Of course, each of these operations give results in V, but closure is about whether they stay in W.

Important: If $\vec{0}$ is not in W, it is not a subspace! Check this first.

<u>Proof</u> for $\vec{v} \in W$, $0\vec{v} = \vec{0}$, which is in W by closure.

Every subspace is itself a vector space. The 8 props satisfied.

The <u>trivial subspaces</u> of V are $\{\vec{0}\}$ and V.

Two principle kinds of subspaces we study: null and span.

Recall: for $m \times n$ A, $\text{null}(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \subset \mathbb{R}^n$
(homogeneous solutions)

Of course, $\vec{0} \in \text{null}(A)$, since $A\vec{0} = \vec{0}$, so could be a subspace.

Given $\vec{x}, \vec{y} \in \text{null}(A)$,
$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$$
so $\vec{x}+\vec{y} \in \text{null}(A)$, too.

Given $\vec{x} \in \text{null}(A)$ and $c \in \mathbb{R}$,
$$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}$$
so $c\vec{x} \in \text{null}(A)$, too.

Thus, $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

Recall: $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \} \subset \mathbb{R}^m$
(all linear combinations)

Again, $\vec{0}$ in span since $\vec{0} = A\vec{0}$.

For $\vec{x}, \vec{y}$ in span, there are $\vec{X}, \vec{Y} \in \mathbb{R}^n$ with
$$\vec{x} = A\vec{X}$$
$$\vec{y} = A\vec{Y}$$

$\vec{x} + \vec{y} = A\vec{X} + A\vec{Y} = A(\vec{X} + \vec{Y})$, so sum in span

$c\vec{x} = cA\vec{X} = A(c\vec{X})$, so scaled in span.
Thus, span is subspace.

For convenience,

def $\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$, the set of lin. combs of cols of A.

So, $\mathrm{Null}(A)$ and $\mathrm{Col}(A)$ are subspaces obtained from a matrix.

When showing a $W \subset \mathbb{R}^n$ is a subspace, it is easiest to show it is Null/Col of a matrix.

ex $W = \left\{ \begin{bmatrix} 2a+b \\ a \\ 3b \\ a-b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

$W = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \right\}$ so W is a subspace of $\mathbb{R}^4$

ex $W = \{ \vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \}$

$W = \mathrm{Null}(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix})$ so W is a subspace of $\mathbb{R}^4$!

What isn't a subspace?

nonex
i) $\mathbb{R}^2 \not\subset \mathbb{R}^3$
ii) Solutions to $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ not solution
iii) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x \geq 0 \right\}$ $-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ not in set
iv) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| xy \geq 0 \right\}$ closed under scalars, not addition

Null and Col

Let us contrast Null and Col of $m \times n$ A:

Null(A)	Col(A)
subspace of $\mathbb{R}^n$	subspace of $\mathbb{R}^m$
to check $\vec{x} \in Null(A)$, see if $A\vec{x} = \vec{0}$	to check if $\vec{b} \in Col(A)$, see if $A\vec{x} = \vec{b}$ consistent
given by constraint $A\vec{x} = \vec{0}$	given by span
to write as a span, must solve $A\vec{x} = \vec{0}$ in parametric vector form	to write as span, just look at the columns of A.

This last point is our focus: Given a subspace, what is a (minimal) collection of vectors which spans it? Minimal for a "basis"

ex $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

$$Null(A) = Span\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

$$Col(A) = Span\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

Linear transformations

We just generalize our previous definition:

def A linear transformation $T: V \to W$ for V, W vector spaces, is a function satisfying linearity:

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u},\vec{v} \in V$

(b) $T(c\vec{u}) = c\,T(\vec{u})$ for $c \in \mathbb{R}$ and $\vec{u} \in V$

Right now, we care about this as far as an analogy to Null/Span:

def The kernel of linear $T: V \to W$ is

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \} \subset V$$

def The image or range of linear $T: V \to W$ is

$$\operatorname{im} T = \{ T(\vec{v}) \mid \vec{v} \in V \} \subset W$$

(the set of images. also denoted $T(V)$)

Like for null/span, ker/im are subspaces of V/W, resp.

For $T: \mathbb{R}^n \to \mathbb{R}^m$,

$$\ker T = \operatorname{null}([T])$$
$$\operatorname{im} T = \operatorname{col}([T])$$

(transf.) (matrix)

<u>ex</u> Let V be diff'ble functions $\mathbb{R} \to \mathbb{R}$,
$T(f) = f' - 2f$

$$\ker f = \{ f \mid T(f) = 0 \}$$
$$= \{ f \mid f' - 2f = 0 \}$$

so solutions to $f' = 2f$.

that is, $f = ce^{2x}$

thus, $\ker f = \operatorname{Span}\{ e^{2x} \}$.

Relationship to one-to-one and onto

$\ker T = \{\vec{0}\} \iff T$ is one-to-one
$\operatorname{im} T = W \iff T$ is onto

$\ker T = V \iff T$ is $\vec{v} \mapsto \vec{0} \iff \operatorname{im} T = \{\vec{0}\}$