Vector spaces

In what follows, we will extract the essence of $\mathbb{R}^n$, which for us is vector addition and scalar multiplication, and formulate the notion of a vector space. A vector space is a thing which behaves much like $\mathbb{R}^n$. One difference is that vectors will no longer necessarily have entries!

The reason for abstracting $\mathbb{R}^n$ is that math is full of objects which are $\mathbb{R}^n$-like, and many functions between them which are linear-transformation-like.

Examples: $\mathbb{R}^n$, polynomials, continuous functions, set of $m \times n$ matrices $\mathbb{R}^{m\times n}$, solutions to a homogeneous system, electric fields, sound waves, power series, image of a transformation.

The definition we give is much more abstract than you have probably dealt with yet, other than n being a number or f a function. Now, the notion of a vector itself has meaning only relative to some vector space V. The question "so what *is* a vector, really?" is simply meaningless.

real as in $\mathbb{R}$. later, complex with $\mathbb{C}$

def A (real) vector space V is three things together:

- a set of vectors in V (we use V to refer to both the space and this set. This is a figure of speech: metonymy.)
- an operation $+ : V \times V \to V$ called vector addition
- an operation $\mathbb{R} \times V \to V$ called scalar multiplication

along with a collection of properties: for $\vec{u}, \vec{v}, \vec{w} \in V$, $c, d \in \mathbb{R}$,

(a) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (b) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (c) there is a $\vec{0} \in V$ with $\vec{v}+\vec{0}=\vec{v}$ for all $\vec{v}\in V$.

(d) There is a $-\vec{u} \in V$ with $\vec{u}+(-\vec{u}) = \vec{0}$ (e) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ (f) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

(g) $c(d\vec{u}) = (cd)\vec{u}$ (h) $1\vec{u} = \vec{u}$.

To reiterate, a vector space V is an object with a set (also called V) and two operations, which satisfy a number of $\mathbb{R}^n$-like properties.

An important principle is "duck typing": If it walks like a duck and quacks like a duck, it's a duck. Here, if a set has an addition like a vector space and a scalar multiplication like a vector space, it's a vector space.

We can use these properties (traditionally called <u>axioms</u>) to show certain facts about <u>all</u> vector spaces.

<u>ex</u> There is only one zero vector. If $\vec{0}, \vec{0}' \in V$ refer to zero vectors, $\vec{0} \overset{(c)}{=} \vec{0}' + \vec{0} \overset{(a)}{=} \vec{0} + \vec{0}' \overset{(c)}{=} \vec{0}'$, so $\vec{0} = \vec{0}'$.

<u>ex</u> $0\vec{u} = \vec{0}$. $0\vec{u} = (0+0)\vec{u} \overset{(f)}{=} 0\vec{u} + 0\vec{u}$ by (d), add $-(0\vec{u})$ to both sides: $\vec{0} = 0\vec{u}$.

<u>ex</u> $-\vec{u} = (-1)\vec{u}$. $\vec{u} + (-1)\vec{u} \overset{(h)}{=} 1\vec{u} + (-1)\vec{u} \overset{(f)}{=} (1-1)\vec{u} = 0\vec{u} = \vec{u}$

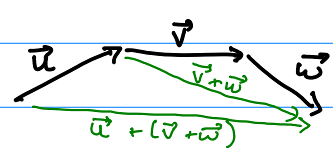
This actually implies $-\vec{u}$ is unique!

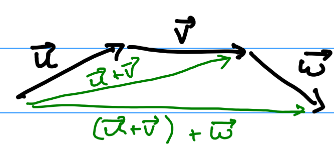
Here are some concrete examples of vector spaces:

- $\mathbb{R}^n$, of course — this is what was abstracted! It doesn't hurt to test ideas against $\mathbb{R}^3$ to check reasonableness.
- Let $\mathcal{A}$ be the collection of all arrows in 3D space, where two arrows are equivalent if they are translated versions of each other (ie., same length and direction). Addition is by the parallelogram rule, scalar by scaling.

<u>ex</u> $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

zero vector?





ex S = set of infinite sequences

$$a = (1, 1, 2, 3, 5, \ldots)$$
$$b = (0, 1, 1, 2, 3, \ldots)$$
$$c = (0, 0, 1, 1, 2, \ldots)$$
$$a - b - c = (1, 0, 0, 0, 0, \ldots) = \mathbb{1}$$

so dependence: $a - b - c - \mathbb{1} = 0$

$(0, 0, \ldots)$ is zero vector

$$-a = (-1, -1, -2, -3, \ldots)$$

ex $\mathbb{P}$ = set of polynomials with $\mathbb{R}$ coefficients
can add polynomials and scale them. (ignoring multiplication of polys)
This is basically just S above (they are "isomorphic")
$\mathbb{P}_n$ = set of polys of degree at most n.

ex Let $\mathcal{U}$ be a set, and V = functions $\mathcal{U} \to \mathbb{R}$ ($\mathcal{U}$ could be the surface of the earth, and $f \in V$ could map positions to temperature). Define $f+g$ and cf by

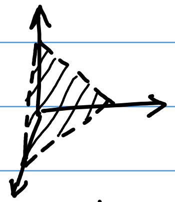
$$(f+g)(t) = f(t) + g(t) \qquad (cf)(t) = c f(t) \qquad (\text{"pointwise"})$$

for $t \in \mathcal{U}$.

$t \mapsto 0$ is the zero vector, $(-f)(t) = -f(t)$.

If $\mathcal{U}$ = time, and f, g are pressure waves, $f+g$ is the result of both sounds occuring simultaneously, and cf is amplification. <u>Interference</u> has to do with linear combinations of waves!

ex Let $S = \{ (x,y,z) \mid x,y,z \in (0,1) \text{ and } x+y+z=1 \}$



a triangle's interior

define $c(x,y,z) = \left(\frac{x^c}{x^c+y^c+z^c}, \frac{y^c}{x^c+y^c+z^c}, \frac{z^c}{x^c+y^c+z^c} \right)$

$$(x,y,z)+(a,b,c) = \left(\frac{ax}{ax+by+cz}, \frac{by}{ax+by+cz}, \frac{cz}{ax+by+cz} \right)$$

the zero vector is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

since $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + (x,y,z) = \left(\frac{\frac{1}{3}x}{\frac{1}{3}(x+y+z)}, \frac{\frac{1}{3}y}{\frac{1}{3}(x+y+z)}, \frac{\frac{1}{3}z}{\frac{1}{3}(x+y+z)} \right)$
$= (x,y,z)$

and $0(x,y,z) = \left(\frac{x^0}{x^0+y^0+z^0}, \dots \right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$

(these are probability distributions — somehow related to entropy)

The next important kind of example of a vector space is one which resides in another as a subset. These tend to be simpler to verify for determining vector-space-hood.

def A <u>subspace</u> W of a vector space V is a subset of V with three properties:
(a) The zero vector of V is in W.
(b) For $\vec{u}, \vec{v} \in W$, $\vec{u} + \vec{v} \in W$ (closure under addition)
(c) For $\vec{u} \in W$ and $c \in \mathbb{R}$, $c\vec{u} \in W$ (closure under scalar mult.)

A subspace is a vector space in its own right, with addition and scalar mult. inherited from V. The required properties are also inherited from V.

Note $-\vec{u} \in W$ since $-\vec{u} = (-1)\vec{u} \in W$ by (c).

ex $V \subset V$ is a subspace.

ex $\{\vec{0}\} \subset V$ is a subspace. Zero subspace

ex $\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.

(a) $\vec{0} \in \text{null}(A)$ since $A\vec{0} = \vec{0}$

(b) for $\vec{x}, \vec{y} \in \text{null}(A)$, $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$

so $\vec{x} + \vec{y} \in \text{null}(A)$

(c) for $c \in \mathbb{R}$ & $\vec{x} \in \text{null}(A)$, $A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}$

so $c\vec{x} \in \text{null}(A)$.

ex $\text{Span}\{\vec{u}, \vec{v}\} \subset \mathbb{R}^n$ is subspace.

(a) $\vec{0} = 0\vec{u} + 0\vec{v}$

(b) for $\vec{x}, \vec{y}$ in span,

$$\vec{x} = c_1\vec{u} + c_2\vec{v}$$
$$\vec{y} = d_1\vec{u} + d_2\vec{v}$$
$$\vec{x} + \vec{y} = (c_1+d_1)\vec{u} + (c_2+d_2)\vec{v} \in \text{Span}\{\vec{u}, \vec{v}\}$$

(c) $\vec{x} = c_1\vec{u} + c_2\vec{v}$

$d\vec{x} = dc_1\vec{u} + dc_2\vec{v} \in \text{Span}\{\vec{u}, \vec{v}\}$.

ex $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

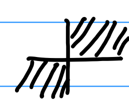
ex Solutions to $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not a subspace of $\mathbb{R}^2$

since $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a solution.

ex $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \mid x \geq 0\right\}$ not a subspace

since $-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin$ set

ex $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \mid xy \geq 0\right\}$ not a subspace



has $\vec{0}$, closed under scalars, but

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, not in set.

Span and null are very important examples.

- $W = \text{Span}\{\cdots\}$ is a subspace generated by the vectors
- null A is a subspace specified by constraints.

intersection. elements in both $\mathcal{U}$ and W are in this

ex $\mathcal{U}, W$ subspaces of V. $\mathcal{U} \cap W$ is a subspace.

(a) $\vec{0} \in \mathcal{U}$ and $\vec{0} \in W$, so $\vec{0} \in \mathcal{U} \cap W$

(b) if $\vec{x}, \vec{y} \in \mathcal{U} \cap W$. Then $\vec{x}, \vec{y}$ in both.

$\vec{x} + \vec{y} \in \mathcal{U}$ and $\vec{x} + \vec{y} \in V$, so $\vec{x} + \vec{y} \in \mathcal{U} + V$.

(c) similar.

For instance, for two planes through $\vec{0}$ in $\mathbb{R}^3$, their intersection is a line through $\vec{0}$.

ex For $\mathcal{U}, W \subset V$ subspaces, not necessarily $\mathcal{U} \cup V$ (union) subspace

Let $\mathcal{U}$ = x-axis, V = y-axis of $\mathbb{R}^2$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \mathcal{U} \cup V.$$

ex For $\mathcal{U}, W \subset V$ subspaces, $\mathcal{U} + V = \{\vec{x} + \vec{y} \mid \vec{x} \in \mathcal{U} \text{ and } \vec{y} \in V\}$ is a subspace (and contains $\mathcal{U} \cup V$. This is the "smallest" subspace containing the union).

(a) $\vec{0} = \vec{0} + \vec{0} \in \mathcal{U} + V$ (b)(c) exercises

$$\mathrm{Span}\{\vec{u}, \vec{v}\} = \mathrm{Span}\{\vec{u}\} + \mathrm{Span}\{\vec{v}\}.$$

Subset notations

$X \subset Y$ "every element of X is in Y. X might equal Y"

$X \subsetneq Y$ "same, but $X \neq Y$. That is, Y contains an element not in X."

Beware: Some people use $\subseteq$ and $\subset$ instead (analogous to $\leq$ and $<$). I don't.