Determinants

Last time: computing determinants via row/column expansion and via row echelon form.

ex $\begin{vmatrix} 2 & 1 & 1 \\ & 3 & 1 \\ & & 5 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ & 5 \end{vmatrix} = 2\cdot3\cdot5$ (upper triangular: product of diagonal entries)

ex $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -6 \end{vmatrix} = -6\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -6$

Two important determinant rules:

1) $\det(A^T) = \det(A)$

This is because column expansions down A^T are row expansions across A

* 2) $\det(AB) = \det(A)\det(B)$ This is a homomorphism (matrix mult. $\longrightarrow$ real number mult.)

Warning $\det(A+B)$ is not $\det(A) + \det(B)$:

$\det(I_2 + I_2) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$

$\det(I_2) + \det(I_2) = 2$.

The book gives a proof of the relationship between row operations and determinants — it is an induction argument on the size of the matrix, using the cofactor expansion.

Aside: it also gives some insight for what happens to the cofactor matrix after performing a row operation on A.

Assuming this proof, let us try to understand rule 2. The first thing is to understand the elementary matrices for elementary row operations.

An elementary row operation, observe, only acts on individual columns of a matrix: a column is in no way affected by the other columns being operated upon. We can treat them as transformations $\mathbb{R}^n \to \mathbb{R}^n$, and (proof omitted) they are linear, hence have a matrix.

<u>ex</u> $n=4$, $R_2 \leftrightarrow R_3$.

$$\begin{aligned} \vec{e}_1 &\mapsto \vec{e}_1 \\ \vec{e}_2 &\mapsto \vec{e}_3 \\ \vec{e}_3 &\mapsto \vec{e}_2 \\ \vec{e}_4 &\mapsto \vec{e}_4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_2 \to R_2$

$$\begin{aligned} \vec{e}_1 &\mapsto \vec{e}_1 \\ \vec{e}_2 &\mapsto 3\vec{e}_2 \\ \vec{e}_3 &\mapsto \vec{e}_3 \\ \vec{e}_4 &\mapsto \vec{e}_4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 - 2R_2 \to R_4$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \\ d-2b \end{bmatrix} \qquad \begin{aligned} \vec{e}_1 &\mapsto \vec{e}_1 \\ \vec{e}_2 &\mapsto \vec{e}_2 - 2\vec{e}_4 \\ \vec{e}_3 &\mapsto \vec{e}_2 \\ \vec{e}_4 &\mapsto \vec{e}_4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

So, if $A \sim B$ by a step of an elementary row operation, $EA = B$, for E the corresponding elementary matrix.

The determinants of elementary row matrices are easy to compute since they are all one step from rref.

swap: $\det = -1$ scale by k: $\det = k$ replacement: $\det = 1$.

So, $A \sim rref(A)$ means $rref(A) = E_k E_{k-1} \cdots E_2 E_1 A$ for $E_1, \cdots, E_k$ elementary matrices, each corresponding to one step of row reduction. It follows that

$$\det A = \frac{\det(rref(A))}{\det(E_1) \cdots \det(E_k)}$$

where $\det(rref(A))$ is 0 or 1.

As for det of product:

Case I: $\det(A) = 0$. By rref correspondence, A is not invertible, so has fewer than n pivots, hence AB has fewer than n as well, so $\det(AB) = 0$. $0 = 0$, so $\det(AB) = \det(A)\det(B)$.

Case II: $\det(A) \neq 0$. Then $A \sim I_n$, so $A = E_k E_{k-1} \cdots E_1$.

$$\begin{aligned}
\det(AB) &= \det(E_k \cdots E_1 B) \\
&= \det(E_k)\det(E_{k-1} \cdots E_1 B) \\
&= \cdots = \det(E_k) \cdots \det(E_1)\det(B) \\
&= \det(E_k) \cdots \det(E_2 E_1)\det(B) \\
&= \cdots = \det(E_k \cdots E_1)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

We deconstructed and reconstructed A because we could take advantage of multiplying by an elementary matrix being the same as performing the row operation, and the effect on det is $\det(EA) = \det(E)\det(A)$.

I like elementary matrices because they "save up" multiple elementary row operations. And: now we see invertible matrices <u>are</u> a sequence of row operations!

In fact, if A has n pivots, $A \sim I_n$, so $E_k \cdots E_1 A = I_n$. Parenthesize carefully: $E_k \cdots E_1$ is A^{-1}! (further justification for $[A \mid I_n] \sim [I_n \mid A^{-1}]$.)

Column operations

Using $\det(A^T) = \det(A)$, we can consider column operations and their effect on $\det A$. For instance,

$$\det(\vec{a}_1 \cdots c\vec{a}_i \cdots \vec{a}_n) = c\det(\vec{a}_1 \cdots \vec{a}_n)$$

or $\det(\vec{a}_1 \cdots \vec{a}_i \cdots \vec{a}_j \cdots \vec{a}_n) = -\det(\vec{a}_1 \cdots \vec{a}_j \cdots \vec{a}_i \cdots \vec{a}_n)$.

There is also a linearity property obtained through cofactor expansion:

$$\det(\vec{a}_1 \cdots \vec{a}_i + \vec{b}_i \cdots \vec{a}_n) = \det(\vec{a}_1 \cdots \vec{a}_i \cdots \vec{a}_n) + \det(\vec{a}_1 \cdots \vec{b}_i \cdots \vec{a}_n)$$

These are actually defining properties for det, along with $\det I_n = 1$.

det is called an $\underset{\text{(swap gives negative)}}{\text{alternating}}$ $\underset{\text{(props 1 and 2)}}{\text{multilinear}}$ $\underset{\text{(takes } n \text{ vectors, the columns)}}{n\text{-form}}$

ex $\begin{vmatrix} 1 & 2 \\ & 3 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 2 \\ & \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ & 3 \end{vmatrix}}_{\text{(same 1st column)}} = 0 + 3\begin{vmatrix} 1 & \\ & 1 \end{vmatrix} = 3.$

Cramer's Rule

Now to learn an extremely inefficient method to solve $[A \mid \vec{b}]$, but for a theoretical purpose: it allows us to understand how $A^{-1}\vec{b}$ changes with respect to both A and $\vec{b}$.

A single-use notation: let $A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{a}_{i-1}\ \vec{b}\ \vec{a}_{i+1} \cdots \vec{a}_n]$ (replace column i of A with $\vec{b}$). We first see that

$$A\, I_i(\vec{x}) = A[\vec{e}_1 \cdots \vec{x} \cdots \vec{e}_n] = [A\vec{e}_1 \cdots A\vec{x} \cdots A\vec{e}_n] = A_i(A\vec{x}).$$

If $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$, then $= A_i(\vec{b})$.

Thus, by the multiplicative rule of det,

$$\det(A)\det(I_i(\vec{x})) = \det(A_i(\vec{b}))$$

For this second determinant,

$$\det(I_i(\vec{x})) = \begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & & x_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_i & \cdots & 0 \\ \vdots & & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{vmatrix} = x_i |I_{n-1}| = x_i$$

(along row i)

We have thus derived

<u>Thm (Cramer's Rule)</u> For invertible $n \times n$ A and $\vec{b} \in \mathbb{R}^n$, the solution $\vec{x} \in \mathbb{R}^n$ to $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}.$$

Note: the complexity of computing the solution $\vec{x}$ using rref is just about that of computing $\det A$, so this is unlikely to be a go-to. However, we can compute x_i for <u>varying</u> A, which is difficult to deal with in row reduction.

<u>ex</u> $A = \begin{pmatrix} 1 & 2 \\ c & 3 \end{pmatrix}$, for $c \in \mathbb{R}$ varying. For $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we compute a solution:

$$x_1 = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ c & 3 \end{vmatrix}} = \frac{1}{3-2c} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ c & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ c & 3 \end{vmatrix}} = \frac{1-c}{3-2c}$$

Of course, we know $A^{-1} = \frac{1}{3-2c}\begin{pmatrix} 3 & -2 \\ -c & 1 \end{pmatrix}$, and this is $A^{-1}\vec{b}$.

ex $\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = 4 \end{cases}$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-7}{-2} = 7/2$$

$$x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{1}{-2} = -1/2$$

(Cramer's rule is actually probably easier than rref for 2×2, or when we start using $\mathbb{C}$ instead of $\mathbb{R}$)

Remember: Cramer's rule only applies when solutions are <u>unique</u>.

A formula for A^{-1}

We calculated that column j of A^{-1} is the solution to $A\vec{x} = \vec{e}_j$. With Cramer's rule, we obtain that entry (i,j) of A^{-1} is

$$\frac{\det A_i(\vec{e}_j)}{\det A}$$

It is easy to expand along column i of $A_i(\vec{e}_j)$, since it has only one 1 in that column, at entry (j,i). This gives $\det(A_i(\vec{e}_j)) = (-1)^{i+j}\det(A_{ji})$, where A_{ji} once again denotes the minor obtained from A by deleting row j and column i, which is $(n-1)\times(n-1)$.

Let $C_{ij} = (-1)^{i+j}\det(A_{ij})$ once again denote the (i,j)-cofactor of A. Then,

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

This matrix of cofactors is the transpose of the cofactor matrix from yesterday. In this form (transposed) it is called the adjugate or classical adjoint of A, denoted adj A.

So, $$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$\left(\text{or} \quad \operatorname{adj}(A)\, A = \det(A)\, I_n\right)$$

ex $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}^{-1}$

cofactors:

$\begin{vmatrix} 2 & 0 \\ 5 & 3 \end{vmatrix} = 6 \qquad -\begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} = -12 \qquad \begin{vmatrix} 4 & 2 \\ 0 & 5 \end{vmatrix} = 20$

$-\begin{vmatrix} 0 & 0 \\ 5 & 3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 \qquad -\begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = -5$

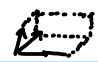
$\det = 1 \cdot 2 \cdot 3$

$\begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} = 0 \qquad -\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2$

so inverse is $\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ -12 & 3 & 0 \\ 20 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1/2 & 0 \\ 10/3 & -5/6 & 1/3 \end{bmatrix}$

(checked that $A^{-1}A = I_3$)

det and area/volume

The columns of a matrix can be interpreted as framing a parallelogram or parallelepiped.  or 
Each elementary row operation has a clear effect on the corresponding volume: scaling scaling, swapping negating, and replacement a volume preserving shear. Thus: $\det(\vec{a}_1, \cdots \vec{a}_n)$ is area/volume! For non-parallepipeds, they are limits of small parallelepipeds, so $\det(A)$ measures the ratio of area/volume change. Application: area of ellipse.