The invertible matrix theorem

Let A be $n \times n$. The following are equivalent:

(a) A is invertible
(b) $A \sim I_n$
(c) A has n pivots
(d) $A\vec{x} = \vec{0}$ has only trivial solution
(e) Columns of A are linearly independent
(f) $\vec{x} \mapsto A\vec{x}$ is one-to-one
(g) A has left inverse
(h) For each $\vec{b} \in \mathbb{R}^m$, $[A \mid \vec{b}]$ solvable
(i) The columns of A span $\mathbb{R}^m$
(j) $\vec{x} \mapsto A\vec{x}$ is onto
(k) A has a right inverse
(l) A^T is invertible

(d)–(g): pivot in every column

(h)–(k): pivot in every row

This is merely a rehashing of our two theorems from before, though they are now tied together since A is square.

One addition is invertibility of A. A has a pivot in every row iff $A\vec{x} = \vec{b}$ is always solvable. If so, $A\vec{v}_i = \vec{e}_i$ can be solved for each i, giving an inverse $A^{-1} = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$. Conversely, an inverse yields solutions $\vec{x} = A^{-1}\vec{b}$.

"iff" means "if and only if"

$(A^T)^{-1}$ is just $(A^{-1})^T$. To check: $(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$.
(so this is inverse pair)

ex $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ has 1 pivot $\Rightarrow$ no inverse

<u>ex</u> $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is invertible

<u>ex</u> If A, B $n \times n$ and AB invertible,
$AB\vec{x} = \vec{b}$ is always solvable $(\vec{x} = (AB)^{-1}\vec{b})$
so $A\vec{y} = \vec{b}$ is always solvable $(\vec{y} = B(AB)^{-1}\vec{b})$
so A is invertible.

$(AB)^T$ is invertible. This is $B^T A^T$, so by a similar argument, B^T is invertible, thus B is too.
Conversely, for A, B invertible $n \times n$, AB is invertible.
This is simple: $B^{-1}A^{-1}AB = I_n$, so

$$\ast \quad (AB)^{-1} = B^{-1}A^{-1}$$

<u>ex</u> Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. <u>If</u> they span $\mathbb{R}^n$, $[\vec{v}_1 \cdots \vec{v}_k]$ has a pivot in every row, so $k \geq n$. If they are indep., the matrix has a pivot in every column, so $k \leq n$.
<u>If</u> span $\mathbb{R}^n$ and indep., $k = n$, and $[\vec{v}_1 \cdots \vec{v}_k]$ is invertible.

The converse is also true.

Recall: the <u>converse</u> of "if P then Q" is "if Q then P"

<u>Determinants</u>

We now concern ourselves with a square matrix A $(n \times n)$.
The determinant, denoted $|A|$ or $\det A$, is a real number which serves as a criterion for invertibility:

$$|A| \neq 0 \iff A \text{ is invertible}$$ (with all consequences of the invertible matrix theorem)

History: by 1700s, mathematicians could compute determinants of systems. ~1812 Cauchy finally defines them a more modern way. 1850 Sylvester defines matrices and shows how to compute determinants with them.

For 2×2 matrices, we saw $a_{11}a_{22} - a_{21}a_{12}$ controls whether $A = [a_{ij}]_{ij}$ is invertible. Can we generalize?

For a given n, we could try to compute closed-form expressions for A^n, seeing if a determinant expression drops out. $n=3$ is given in the textbook by this method. For $n=1$, it is easy: $\det[a] = a$.

Here is the product of hundreds of years of toil:

<u>Laplace expansion</u> The determinant of A can be computed by "expanding along the first row":

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - \cdots$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

where A_{ij} is a <u>minor</u> (not entry (i,j) in this context!) given by deleting row i and column j.

Well, that is how to compute a determinant. It is recursive, so at each step, $n-1$ more determinants must be computed!

The number of multiplications to compute this is given by $T(n) = (n-1)\,T(n-1) + n-1$, with $T(1) = 0$. This gives $T(n) = (n-1)!$, which is absurd. It is fine for small matrices, but we will have better methods for larger ones.

ex For 2×2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \begin{vmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{vmatrix} - b \begin{vmatrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{vmatrix} = ad - bc$$

Do not forget the expansion is an alternating sum.

ex
$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1\cdot \begin{vmatrix} \cancel{1} & \cancel{5} & \cancel{0} \\ \cancel{2} & 4 & -1 \\ \cancel{0} & -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} \cancel{1} & \cancel{5} & \cancel{0} \\ 2 & \cancel{4} & -1 \\ 0 & \cancel{-2} & 0 \end{vmatrix} + 0\cdot \begin{vmatrix} \cancel{1} & \cancel{5} & \cancel{0} \\ 2 & 4 & \cancel{-1} \\ 0 & -2 & \cancel{0} \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= \big(4\cdot 0 - (-1)(-2)\big) - 5\big(2\cdot 0 + (-1)\cdot 0\big)$$

$$= -2.$$

ex
$$\begin{vmatrix} 1 & & \\ 2 & 2 & \\ 3 & 3 & 3 \end{vmatrix} = 1\cdot \begin{vmatrix} \cancel{1} & & \\ \cancel{2} & 2 & \\ \cancel{3} & 3 & 3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} \cancel{2} & \\ \cancel{3} & 3 \end{vmatrix} = 6$$ (product of diagonal for triangular matrix).

The cofactor matrix of A is given by

$$\left[(-1)^{i+j}\det(A_{ij})\right]_{ij}$$

Entry (i,j) of this is the (i,j)-cofactor of A. We will write this as $C_{ij} = (-1)^{i+j}\det(A_{ij})$ just for now. The $(-1)^{i+j}$ is there to make a checkerboard pattern of $+$ and $-$:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We give without proof another way to compute determinants.

Cofactor expansion across row i is given by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots = \sum_{k=1}^{n} a_{ik}C_{ik}$$

and down column j by

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

The implicit theorem is that all three methods give the same result. At least across row 1 is Laplace expansion.

ex
$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 0\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = -2.$$

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = -(5)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= -5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= -5(0) + 4(0) + 2(-1) = -2.$$

We used $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ pattern for signs.

Determinants and row operations

The following four properties give a much better method to compute determinants. Let A be $n \times n$.

1. If B is A after scaling a row by k,
   $\det B = k \det A$
2. If B is A after swapping two rows,
   $\det B = -\det A$
3. If B is A after a replacement operation,
   $\det B = \det A$
4. If A is upper triangular (i.e., A has only zeros below the diagonal), $\det A$ is the product of the diagonal entries.

ex
$$\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \quad \tfrac{1}{2}R_1 \to R_1 \quad \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$$\begin{matrix} R_2 - 3R_1 \to R_2 \\ R_3 + 3R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{matrix} \quad \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad R_4 - \tfrac{1}{2}R_3 \to R_4 \quad \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The det. of this last matrix is $1 \cdot 3 \cdot (-6) \cdot 1 = -18$

of third, -18

of second, -18

of first, $\frac{1}{1/2}(-18) = \boxed{-36}$

since $\det B = \frac{1}{2} \det A$,

so $\det A = 2 \det B$.

This method takes, worst case, $\frac{n(n+1)(2n+1)}{6} + 2(n-1)$ $\approx \frac{2n^3}{6}$ multiplications. Much better than factorial!

The break-even point appears to be $n = 6$ though:

n	$(n-1)!$	$\frac{n(n+1)(2n+1)}{6} + 2(n-1)$
2	1	7
3	2	18
4	6	36
5	24	63
6	120	101
7	720	152
8	5040	218

Note: these are *worst* case. Your particular matrix may take much fewer operations.