Matrix multiplication (reprise)

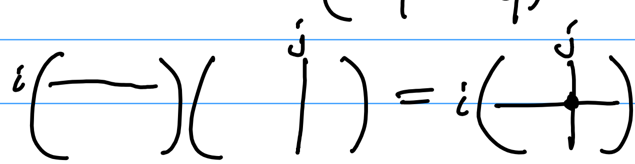
The product of two matrices A and B is completely characterized by the rule

$$(AB)\vec{v} = A(B\vec{v}).$$

The way this works is simple: The ith column of a matrix M is $M\vec{e}_i$ (with $\vec{e}_i$ all zero, but a 1 in entry i), so substituting $\vec{v} = \vec{e}_i$, we have $(AB)\vec{e}_i = A(B\vec{e}_i)$. That is,

column i of AB equals $A\vec{b}_i$
(with $\vec{b}_i$ = column i of B)

ex
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \left(\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix}\right) \quad \left(\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}3\\4\end{pmatrix}\right) \right)$$
$$= \begin{pmatrix} 4 & 8 \\ 4 & 9 \end{pmatrix}.$$

or use  rule:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 0 + 3\cdot 1 & 1\cdot 0 + 2\cdot 1 + 3\cdot 2 \\ 0\cdot 1 + 1\cdot 0 + 4\cdot 1 & 0\cdot 0 + 1\cdot 1 + 4\cdot 2 \end{pmatrix}$$

The rule can also be given entrywise:

$$(AB)_{ij} = \sum_k \underset{\substack{\uparrow \\ \text{row fixed}}}{A_{ik}} \underset{\substack{\uparrow \\ \text{column fixed}}}{B_{kj}}$$

Transposes

An $m \times n$ matrix A has an $n \times m$ **transpose** A^T which is A "flipped" over its diagonal. $(A^T)_{ij} = A_{ji}$

ex $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

Properties
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(rA)^T = rA^T \quad (r \in \mathbb{R})$
- $(AB)^T = B^T A^T$ ← reversed!

This last prop. is interesting. It is not too hard to show:

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$
$$= \sum_k B_{ki} A_{jk}$$
$$= \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$$

Corresponding entries are equal, so $(AB)^T = B^T A^T$.

Transposes will show up later in orthogonality.

Identity matrices $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $(n \times n)$ have the following properties:
Say A is $m \times n$.
- $AI_n = A$
- $I_m A = A$

They are the equivalent of $1 \in \mathbb{R}$ under multiplication.

The inverse of a matrix

Like for numbers, sometimes we can multiply by a matrix to undo a previous multiplication. Unlike numbers, AB and BA might not equal, so there are two kinds of inverses.

Let A be $m \times n$ and C be $n \times m$.

- C is a left inverse if $CA = I_n$

  ex $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

  Since I_n has $\begin{cases} \text{a pivot in every row, so must } C \\ \text{a pivot in every column, so must } A \end{cases}$

- C is a right inverse if $AC = I_m$

  ex same but reverse roles of matrices.

- C is an inverse if $AC = CA = I_n$ (need $n = m$)
  (also two-sided inverse)

Correspondence:
- A has left inverse $\Leftrightarrow$ $\vec{x} \mapsto A\vec{x}$ is one-to-one
- A has right inverse $\Leftrightarrow$ $\vec{x} \mapsto A\vec{x}$ is onto
- A has both $\Leftrightarrow$ $\vec{x} \mapsto A\vec{x}$ is one-to-one and onto (bijective)

Let's just think about square A with inverse today.
If A has inverses C, D, $C = CI_n = CAD = I_nD = D$, so inverse, if it exists, is unique. Denote it by A^{-1} (if exists, otherwise singular).

1st method of computation

Since $A^{-1}A = I_n$, we have $A^{-1}A\vec{v} = \vec{v}$. If we solve $A\vec{v}_i = \vec{e}_i$, for $\vec{v}_i$, then $A^{-1}A\vec{v}_i = \vec{v}_i \iff A^{-1}\vec{e}_i = \vec{v}_i$. That is,

$*$ column i of A^{-1} is the solution to $A\vec{x} = \vec{e}_i$.

This suggests computing solutions $[A \,\vdots\, \vec{e}_i]$ repeatedly to build A^{-1}.

Do we really want to do rref n times? Observe: row operations only operate on rows! May as well compute rref of $[A \,\vdots\, \vec{e}_1, \vec{e}_2 \cdots \vec{e}_n] = [A \,\vdots\, I_n]$ once. Since A must have n pivots, the right side of rref will be A^{-1}, as in $[I_n \,\vdots\, A^{-1}]$.

ex.
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}}_{A^{-1}}$$

Check:
$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1(-3)+2(2) & 1(2)+2(-1) \\ 2(-3)+3(2) & 2(2)+3(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

ex.
$$\left[\begin{array}{cc|cc} 1 & x & 1 & \\ 0 & 1 & & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -x \\ 0 & 1 & 0 & 1 \end{array}\right]$$

so
$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}.$$

2nd method: A is 2×2

* For $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$

if $ad-bc\neq 0$ (the "determinant")

This is useful and easy enough to memorize ("swap major, negate minor, divide by determinant").

To prove, we need only check it's an inverse!

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}=\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc\end{pmatrix}$$

so when divided by det., $= I_2$.

ex $R_\theta=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ is rotation θ CW

$$R_\theta^{-1}=\underbrace{\frac{1}{\cos^2\theta+\sin^2\theta}}_{=1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$

and $\cos\theta=\cos(-\theta)$
$-\sin\theta=\sin(-\theta)$

so $=\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}=R_{-\theta}$

rot. by θ CCW,

Intuitively the inverse.

If A has an inverse, $A\vec{x}=\vec{b}$ always has $\vec{x}=A^{-1}\vec{b}$ as soln, so A has a pivot in every row. A is square, so pivot in every column, too. Thus, $A^{-1}\vec{b}$ is the unique solution.

Properties

- $(A^{-1})^{-1} = A$. This is because $AA^{-1} = I_n$ and $A^{-1}A = I_n$, so A is inverse of A^{-1} (denoted $(A^{-1})^{-1}$).
- $(AB)^{-1} = B^{-1}A^{-1}$. This is because $(B^{-1}A^{-1})AB = B^{-1}I_nB = I_n$, $AB(B^{-1}A^{-1}) = AI_nA^{-1} = I_n$. So $B^{-1}A^{-1}$ plays role of inverse to AB.
- $(A^T)^{-1} = (A^{-1})^T$. $(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$, $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$, so A^T is inverse of $(A^{-1})^T$.

(Sometimes people write these as A^{-T} because of the rule)

Invertible matrix theorem

A is $n \times n$.

(all true or all false)

The following are equivalent:

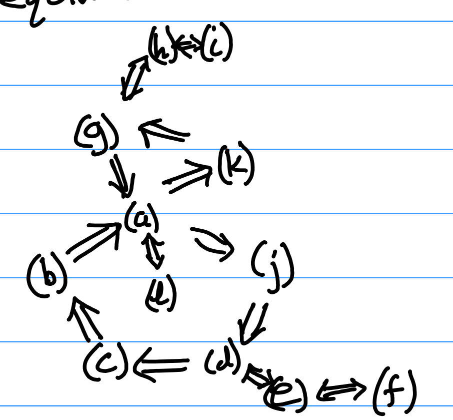
(a) A is invertible
(b) $A \sim I_n$
(c) A has n pivots
(d) $A\vec{x} = \vec{0}$ has only trivial solution
(e) Columns of A are linearly independent
(f) $\vec{x} \mapsto A\vec{x}$ is one-to-one
(g) For each $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is solvable.
(h) The columns of A span $\mathbb{R}^n$
(i) $\vec{x} \mapsto A\vec{x}$ is onto
(j) A has a left inverse.
(k) A has a right inverse.
(l) A^T is invertible

Caution

A is square

Pivot in every row equiv. to one group of statements
Pivot in every col equiv. to another
Pivotal: rows = cols in square matrix. This joins them.

Book's implications for equivalence.



ex $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is not invertible.

ex $\begin{pmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{pmatrix}$ is invertible

ex If A, B $n \times n$ and AB is invertible,
$AB\vec{x} = \vec{b}$ is always solvable $(\vec{x} = (AB)^{-1}\vec{b})$
so $A\vec{y} = \vec{b}$ is always solvable $(\vec{y} = B(AB)^{-1}\vec{b})$
so A is invertible.

$(AB)^T$ is invertible, which is B^TA^T. By similar argument, B^T is invertible, so B is invertible.

Behold the power of the invertible matrix theorem!

ex If $\vec{v}_1, \dots, \vec{v}_k$ are vectors which span $\mathbb{R}^n$
$[\vec{v}_1 \cdots \vec{v}_k]$ pivot in each row $(k \geq n)$

if indep., $[\vec{v}_1 \cdots \vec{v}_k]$ pivot in each col $(k \leq n)$

if both, $k = n$, and $[\vec{v}_1 \cdots \vec{v}_n]$ is invertible.